The mathematical and philosophical foundations of quantum field theory.

Saunders, Simon Wolfe

THE MATHEMATICAL AND PHILOSOPHICAL FOUNDATIONS

OF QUANTUM FIELD THEORY

by SIMON WOLFE SAUNDERS

Ph.D. thesis to be submitted to
King's College, University of London
April 1988

THE MATHEMATICAL AND PHILOSOPHICAL FOUNDATIONS OF QUANTUM FIELD THEORY

Abstract

The thesis is primarily concerned with these objectives: to say *what* is a quantum field theory, and to explain *why* and *how* relativistic quantum field theory differs from non-relativistic quantum field theory, even in the free or weakly interacting (quasi-free) case. Following the ideas of Irving Segal, I shall establish that in this case there is an essential identity in structure of the non-relativistic and relativistic field theories. Novel but straightforward applications of this theory are made to the complex scalar field, and in relation to the Dirac hole theory.

Although the structure of the relativistic and non-relativistic quasi-free theories is essentially identical, the concept of localization finds different expressions. This plays a fundamental role when interactions are introduced, and leads to two quite distinct notions of causality.

I shall confine the detailed study to the massive scalar and spin 1/2 *linear* field theories, for the most part in the quasi-free case. Not even the latter are trivial, for they describe the observed phenomenology and are therefore of central empistemological importance to relativistic quantum theory.

I also advance a general interpretative framework for the philosophical analysis of quantum theory. This is essentially a realist interpretation founded on abstract C^*-algebras, and it is applied to the measurement problem.

The physical and mathematical theories that I draw upon are developed in a historical context. The mathematical theory is presented in a largely heuristic way.

This thesis is dedicated to my parents, for their love, encouragement, and support, over all these years.

Acknowledgements

I would like to thank above all John Hill and Mary Lindsay, my guardian angels, and Michael Redhead, Harvey Brown and Paul Tappenden, for their constant encouragement and interest in the thesis. My greatest intellectual debts are to Peter Taylor, who taught me the importance of mathematics in the philosophy of physics, and to Michael Redhead, who never allowed himself to be fooled by it.

To the numerous friends and relations who have put up with me and my work habits all these years - thank you for your forbearance.

I owe a particular debt of gratitude to Harold Saunders, Cathy Jones, Peter Twyman, Joanne Robertson, Michael Green, Graham Segal, and above all, to Wolfson College. To Gunthard Müller, Nigel Rudgewick-Brown, and Didier Delaplace - many thanks for all your help with software and hardware.

To Tessa Hall - thank you for all your encouragement, and for your confidence and support in my endeavours.

The errors and failings of the thesis I have managed to put in unassisted.

CONTENTS

PART 2. MATHEMATICAL AND PHILOSOPHICAL FOUNDATIONS

PART THREE: APPLICATIONS TO THE INTERPRETATION OF QUANTUM FIELD THEORY

Introduction

Physics is currently facing a crisis as grave and far-reaching as any in its turbulent history. What is more this crisis is primarily conceptual: it is the question of the relationship between geometry and quantum theory.

If it is only in quantum gravity that one must face up to this problem in its full and paradoxical complexity, already in the transition from the Galilean to the Lorentz group one has a new richness to the theory that is quite staggering. Quantum field theory, antiparticles, particle creation and annihilation processes - not to mention the hopelessly intractable problem of making sense of perturbation theory - seem to flow from the combination of quantum theory and relativity alone.

It is probably too much to ask that the philosophy of physics can assist in solving the dilemma of the proper relationship of quantum theory to gravity. If the philosophy of physics has any lasting contribution to the great intellectual creations of physics, it is in the realm of exegesis and not heuristics. This being so, the philosophy of physics is above all concerned with the logical structure of physical theory, and with the metaphysical and conceptual dilemmas posed by physics.

It is especially in the context of definitive physical theory - theory which is in some sense fundamental, yet at the same time physically and mathematically well-defined - that its philosophical exegesis may be both relevant and of some enduring value. In relativistic quantum physics there is probably only one such theory - and that is the theory of free or weakly interacting systems. I therefore make no apology that in detailed applications, this thesis is concerned only with the linear case.

It is in this context that I formulate and eventually answer the question: how and why does relativistic quantum field

theory differ from non-relativistic quantum field theory? This problem brings in its wake a host of others: the relationship of NRQFT to NRQM; the proper formulation of RQM and the interpretation of the relativistic wave equations; the difficulties of developing a configuration space Born interpretation in RQM; the particle interpretation of free fields; the relationship of antimatter to negative frequency solutions; and so on. All of these problems are resolved in Part 3, most especially in Section 3.4.

Despite the enormous literature on the interpretation of NRQM, there is not a single text at a specialist level devoted to the history and philosophy of quantum field theory or relativistic quantum theory. In this situation my treatment has a strong historical orientation. At the same time, the history of QFT and RQT is not my main concern, and is used primarily as a methodological device in the elaboration of the basic ideas. I *am* concerned to preserve contact with the rigorous basis of NRQM; for this reason throughout I consider the canonical theory, rather than path integral methods.

In parallel with the analysis of the relationship of the relativistic to the non-relativistic theory, I attempt to formulate a realist interpretation of quantum physics. In this connection I develop a philosophy of the nature of general mathematical descriptions of the world, that is descriptions which are not too closely tied to any specific phenomenology, and a realist account of QFT which is not confined to linear systems. This is developed in Part 2. Part 1 is a historical survey of the development of QFT and RQFT in the period 1925-1935; a number of inter-related difficulties, acutely felt at the time, are there formulated to be solved in Part 3. In this process I shall draw extensively on the mathematical theory developed in Part 2. In Section 3.5 the central new conceptual feature of QFT, the existence of inequivalent representations, is considered in application to the measurement problem.

PART 1: ELEMENTS IN THE HISTORY OF QUANTUM FIELD THEORY

Introduction

There are many texts in the history of quantum theory which bear on the origins of quantum field theory (QFT), but none specifically devoted to this subject. In the space available I shall not provide a historical survey so much as a historical account of what I shall call the canonical theory of the free fields, and the comparison of this theory to the "standard formalism" of relativistic qauntum field theory (RQFT) as defined in the mid-1930's. Dirac's seminal paper of 1927 is, however, discussed in detail, together with the relevant background.

My concern in this thesis is the logical structure of linear field theory and its relationship with the canonical structure and interpretation of non-relativistic quantum mechanics (NRQM). For a variety of reasons linear field theory is usually formulated in the same way as it was in the mid - thirties. Whilst it would be sufficient, for the purposes of this thesis, to simply summarize the standard formalism of RQFT and the canonical structure of NRQFT, in view of this situation - and because one has a convenient bench-mark, from the point of view of interpretation - I shall present this theory in its historical context.

Apart from the general question of the logical relationship between the relativistic and non-relativistic field theories, the interpretational issues that I shall be most concerned with are the wave-particle duality, the Born interpretation, the measurement problem, and the use of abstract methods in quantum physics. All of these issues take on dramatically new dimensions in field theory; it will

be helpful to review the early debates on these topics *in the context of field theory and/or relativistic quantum mechanics.* I shall *not* consider the much more extensive debates on the general interpretation of NRQM, in particular I shall have nothing to say about complementarity or the Bohr-Einstein controversy. This omission might seem curious, and I justify it on three counts: first, this controversy is concerned neither with field theory, relativistic theory, or their relationships to NRQM, second the controversy concerns two conflicting views on the interpretation of physics, complementarity on the one hand, and the classical world view on the other, neither of which are our concern in this thesis, and third, that anyway the area is well-covered in the secondary literature (*e.g.* Jammer [1974], Schlipp [1970]).

The history of QFT properly begins with Planck and the black-body problem, but the early phases of this history are well-known, and the details are not relevant to this thesis. We take up the story in 1925, at which time the heuristic basis of QFT was already laid down. For excellent historical surveys of the early history of quantum theory, I refer to Jammer [1966], Kuhn [1978], and Hund [1974].

1.1. The Origins of Quantum Field Theory: 1925-1926.

> The properties of elementary processes ...make it seem almost inevitable to formulate a truly quantized theory of radiation.
>
> A. Einstein 1917

> ...the fact that xy was not equal to yx was very disagreeable to me. I felt this was the only point of difficulty in the whole scheme, otherwise I would be perfectly happy.....
>
> W. Heisenberg

1.1.1. A note on abstract methods.

I wish to *briefly* describe the reasoning that led to the creation of the matrix mechanics, because in this process we witness the emergence of an altogether new heuristic in the formulation of physical law, which is to say the first introduction of abstract methods into physics[1].

Bohr, in his profound study of the correspondence limit (Bohr [1918]) for multiply periodic systems had shown that for high frequency components in the Fourier expansion of a dynamical variable, the frequencies which entered into the

[1] Is Newton's introduction of infinitesimals into the mathematical description of motion a precursor? The physical reference of infinitesimals was never in doubt, so that this situation does not greatly differ from the early applications of numbers in the description of the world; there too, particularily in the case of real numbers, the mathematical definition lagged centuries behind their physical interpretation. Viewed with such a broad historical perspective, I would claim that both cases essentially use the abstract method. Of course classical analysis has come to be viewed as a concrete body of formalism, and it is in comparison to this that the matrix mechanics first appeared as abstractly defined.

exponent were indeed proportional to the classical difference in energies associated with those harmonics of the motion, in accordance with the Bohr frequency rule. This theory played a pivotal rôle in the dispersion theory, as elaborated by Born [1924], Van Vleck [1924], and Kramers and Heisenberg [1925]. In his great paper of [1925] Heisenberg considered how this correspondence might be preserved for all harmonics in the expansion:

$$x_n(t) = \sum_\alpha q_\alpha(n)\, e^{i\omega(n)\alpha t} \qquad (1)$$

which, for *multiply periodic systems* has the classical interpretation: n defines a fundamental frequency of the system (namely $\omega(n)$) whilst $\alpha \in \mathbb{Z}^+$ (the positive integers) fixes the harmonic of the motion. The difficulty is obviously that for fixed n, the frequency in the exponent is always an integral multiple of the fundamental. Further, no algebraic manipulations can remedy this fact. Consider, for example, the expansion for $x_n^2(t)$:

$$x_n^2(t) = \sum_{\alpha,\beta} q_\alpha(n) q_\beta(n) e^{i\omega(n)(\alpha+\beta)t} . \qquad (2)$$

It was a stroke of great genius that Heisenberg perceived that the problem with the classical theory resided at this fundamental level, in the *algebraic structure of the classical representation of dynamical variables.* At the level of the Fourier coefficients it was indeed not *a priori* obvious that no other possibilities existed; if one writes

$$x_n^2(t) = \sum_\gamma r_\gamma(n) e^{i\omega(n)\gamma t} \qquad (3)$$

then classical analysis leads to the conclusion that the Fourier components combine as:

$$r_\gamma(n) = \sum_\alpha q_\alpha(n) q_{\gamma-\alpha}(n)$$

as follows from the substitution $\alpha+\beta = \gamma$ in Eq.(2) to obtain:

$$x_n^2(t) = \sum_{\gamma,\alpha} q_\alpha(n) q_{\gamma-\alpha}(n) e^{i\omega(n)\gamma t} .$$

If we are to avoid the consequence that only integral multiples of the fundamental frequency are obtained, we must avoid the algebraic combination law

$$\omega(n)\alpha + \omega(n)\beta = \omega(n)(\alpha+\beta). \qquad (4)$$

On the other hand the dependence of the frequencies that occur in the Fourier expansion of multiply periodic motion

on *two* variables is welcome, for these should (by the Bohr frequency condition) refer to two energy levels of the system. Therefore Heisenberg retained the expansion of Eq.(1), but gave up Eq.(4), introducing instead the new notation

$\omega(n)\alpha \longleftrightarrow \omega(n, n-\alpha)$

$q(n)\alpha \longleftrightarrow q(n, n-\alpha)$

and leaving the question of the way in which these quantities are to be algebraically combined open. The trick is obviously to prevent the variable α from acting purely multiplicatively, and perhaps having a more symmetric rôle in comparison with the variable n. It was, I suggest, pure serendipity that he hit on the hypothesis that the ω's are to be combined as:

$\omega(n, n-\alpha) + \omega(n-\alpha, n-\beta) = \omega(n, n-\beta)$

From which it follows (by similar manipulations to the foregoing), that the q's obey the algebra:

$$\sum_{\alpha} q(n, n-\alpha)q(n-\alpha, n-\beta) = r(n, n-\beta). \qquad (5)$$

If this step was serendipitous, it was once again a creative analysis of the highest calibre that enabled Heisenberg to treat the anharmonic oscillator using this algebra, and by which Born and Jordan (and independently Dirac) subsequently systematized the resulting formalism in the form of the matrix mechanics. Eq.(5) defines a *matrix algebra*; in this way Heisenberg formulated the fundamental feature of quantum theory: *dynamical variables associated with microphysical systems obey a non-commutative associative algebra.*

1.1.2. 1925; preliminary attempts at field quantization.

In 1925 a great deal was known about the quantum theory of the radiation field: following the Bothe-Geiger experiment of [1924] and that of Compton and Simon in [1925], refuting the Bohr, Kramers and Slater theory of [1924], the photon had at last entered the mainstream of quantum theory. The Einstein paper of [1917], which had proved so influential in motivating the Ladenberg-Kramers theory of dispersion, the later Bohr theory of the correspondence principle, and thereby the Heisenberg matrix mechanics, provided the

fundamental heuristic of radiative interactions. At the same time the theories of Debye and Ehrenfest, especially as elaborated by de Broglie in his theory of statistics, made clear that in some sense cavity radiation could be mathematically modelled as a system of stationary waves, which take on discrete excitations, corresponding to the number of photons present with the corresponding frequency. If one considered the radiation field as a dynamical system, to be described by the quantization of the classical theory, the problem would appear to be well-posed, both conceptually and mathematically.

Nevertheless the early development of QFT was not motivated in quite this way, and I suggest for this reason there was no attempt to develop a free theory of electromagnetism (QEM) from first principles. That came much later: with the Heisenberg-Pauli theory of [1929], [1930], after the development of NRQFT. The initial concern was rather to establish that the matrix mechanics did, in principle at least, lead to a correlation of the modulus square of matrix elements (interpreted as the probabilities of transitions from one atomic energy level to another) with the intensity of the radiation emitted by this transition. This motivation was explicit in the first quantum mechanical discussion of radiation, to be found in the 4th chapter of Born and Jordan's historic paper of 1925 "On Quantum Mechanics". This chapter, due entirely to Jordan (*cf.* van der Waerden [1967 p.39], began with the statement:

> Following Heisenberg, the square of the absolute value $|q(nm)|^2$ of the (matrix) elements of q for the case of a Cartesian coordinate system determines the transition probabilities. In conclusion, we would like here to specify in what way this assumption can be established from general principles. It is necessary to begin with the question, in what way the fundamental equations of electrodynamics are reinterpreted in terms of the new theory. (Born and Jordan [1925 p.883])

Jordan's strategy was to calculate the Poynting vector for radiative emission from an electric dipole located at the origin, using matrix analogues of the classical equations. The classical fields may be written in a Fourier series:

$$E(x,t) = \sum_i A_i e^{i\omega_i(t-x/c)} \qquad (6)$$

whereas the Heisenberg theory (for periodic systems) proceeded from the expansion:

$$x(t) = \sum_i q_i e^{i\omega_i t}. \qquad (7)$$

Heisenberg [1925] effected the transition to a matrix mechanics by replacing Eq.(7) with the collection of functions:

$$x(t) = \quad q(n,n-\alpha)\ e^{i\omega(n,n-\alpha)t} \qquad (8)$$

or, following the simplification introduced in Chapter 1 of Born and Jordan [1925] simply:

$$x(t) = q(n,m)e^{i\omega(n,m)t}$$

in which $n,m \in \mathbb{Z}$ label the stationary energy levels of the system.

Declaring[2] that it is clear, without a shadow of a doubt, how to treat the radiation field, Jordan did precisely the wrong thing; he proceeded from Eq.(6), amd wrote down the expression:

$$A(n,m)e^{i\omega(n,m)(t-x/c)} \qquad (9)$$

in its place, declaring that "the matrix elements are associated with monochromatic plane waves". That is actually all that he had to say about the radiation field itself, but the implication is clear: the A(n,m)'s are not matrices (as they should be) but matrix elements. Neither did he comment on the question of whether, as in the transition from Eq.(7) to (8), the summation should be dropped in Eq.(9) (it should not), nor did he attempt to establish the commutation relationships obeyed by the A's.

But he made a positive contribution all the same. In line with the comments above, he considered the classical relationship between the Poynting vector and a dipole with coordinates x located at the origin:

$$-\frac{dU}{dt} = \frac{2e^2}{3c^3}\ddot{x}^2 \qquad (10)$$

obtained by expressing the electric and magnetic fields as

[2] Es ist kaum ein Zweifel möglich, wie man dieses System zu behandeln hat.

functions of their source. Defining the time average $\langle \bar{x} \rangle$ as the diagonal matrix, which coincides with the diagonal entries of the matrix x, and determining the latter from Eq.(8), he was able to conclude that:

$$- \langle \frac{dU}{dt} \rangle = \frac{32\pi^4 e^2}{3c^3} \sum_k \omega(n,k)^4 |q(n,k)|^2.$$

In this sense he claimed to have established that Heisenberg's interpretation of the modulus square of the matrix elements was indeed a consequence of the matrix mechanics. We conclude, however, that this analysis has told us nothing about quantum field theory; effectively, we have merely observed that if the classical relationship Eq.(10) remains valid in the quantum theory, then the mean rate of energy loss is indeed proportional to $|q(n,k)|^2$. This result is not trivial, but it is only the smallest of first steps to a QFT.

The Göttingen group returned to the theory of quantum fields in the second contribution, submitted in November of 1925 (Born, Heisenberg and Jordan [1926]), but again the focus of the discussion was indirect: the Einstein fluctuation formula, which had played such a pivotal rôle (Einstein [1909]) in early formulations of the wave-particle dualism[3]. Debye had shortly after ([1910]) shown that the Planck distribution could be obtained on the simple picture that the normal modes of cavity radiation could only carry energy of integral multiples of $h\nu$, but this same hypothesis did not appear to lead to the fluctuation formula (Ornstein and Zernike [1919], Ehrenfest [1925]). It was evidently through discussions with Ehrenfest that the authors had considered this and related problems in quantum statistics[4], and hoped to show that that the new mechanics resolved all these

[3] Statistical considerations had begun the quantum theory; as we see, they also began quantum field theory proper. Indeed, one can view the entire history of quantum theory from 1900 to 1925 as primarily the history of QFT.

[4] Born et al explicitly acknowledge Ehrenfest's influence; the related issue is specifically the additivity of the entropy, which had long been troubling the theory of quantum statistics. See (1.1.4).

difficulties. It was equally evident that they were *not* motivated to formulate a QFT on the basis of the light quantum hypothesis; indeed, in connection with the Debye theory, they remarked:

> Such a mixture of theoretical wave and light-quantum considerations would seem to us hardly to accord with the real nature of the problem. Rather, we believe it to be consistent to separate the theoretical wave-aspect of the problem completely from the theory of light-quanta, that is to say, to treat the wave-statistics of black-body radiation throughout by the more general statistical rules applying *e.g.* to the quantum theory of atomic systems. The statistics applicable to light quanta is then, as we shall show, Bose statistics. This finding hardly seems unnatural, since this statistics has nothing to do with the hypothesis of independent light-corpuscules, but rather to be regarded as carried over from the statistics of eigenvibrations - which just shows that the assumption of statistically independent light-corpuscules would not meet the case correctly.
> However, in each such treatment of cavity radiation by quantum theory hitherto....although it led to Planck's law....it did not yield the correct (fluctuation formula)....this caused us to hope that the modified kinematics...would yield the correct value for the interference fluctuations, thus precluding the above contradictions and opening the possibility of setting up a consistent system of statistics for black-body radiation. (Born *et al* [1926 p.376-377]).

There are a number of issues raised by this passage[5]; I wish only to emphasise that since clearly the *existence* of

[5] The Debye formulation only fails for the fluctuation formula because of the precise statistical assumptions used by Ornstein, Zernike, and Ehrenfest; de Broglie had by this time shown how they were to be modified, namely by assuming Boltzmann statistics for the harmonic oscillators, i.e. regarding all the photons of a single frequency as a single individualizable sysem (see 1.1.4). Concerning the derivation of the Planck distribution from the canonical basis of the matrix mechanics, in fact their arguments differed little from those of Debye. The entire analysis was later criticised by all three authors concerned: by Jordan in [1929], by Heisenberg in [1931], and by Born and Fuchs in [1939]. These criticisms in turn have been shown to be flawed, and as Gonzales and Wergeland [1972] have made clear, the whole issue is conceptually muddled and inherently ambiguous. The Einstein fluctuation formula is a superb example of a theoretical result that once occupied centre place in the development of physics, but that has since come to be perceived as inconclusive and of peripheral interest.

light-quanta was not in doubt, the authors rather objected to the *manner* of their introduction (or association) to the classical theory. One might paraphrase their objections to the Debye treatment along the lines: the light quanta should not be put in "by hand", but should follow from the fundamental mathematical laws of the matrix mechanics, and the statistics of these quanta should likewise follow from these laws.

These ideas lead them to within a hairsbreadth of a fully-fledged QFT; but they stop just short of it. We recall that if $z_\nu d\nu$ is the number of normal modes per unit volume in the frequency range $\{\nu, \nu+d\nu\}$ and E_ν is the associated energy density, then the mean square deviation for the energy in the sub-volume V of an (infinite) radiation field $\langle \Delta E_\nu^2 \rangle = \langle (E_\nu - \langle E_\nu \rangle)^2 \rangle$ follows by standard techniques from the Planck distribution formula:

$$\langle \Delta E_\nu^2 \rangle = h\nu \langle E_\nu \rangle + \langle E_\nu^2 \rangle / z_\nu V. \qquad (11)$$

The second term alone follows from the classical field theory (the "wave" aspect) whilst the first alone follows from the photon theory (using Boltzmann statistics).

Born *et al* proceeded as follows. Let the lateral displacement of the string be u(x,t); introduce the Fourier expansion:

$$u(x,t) = \sum_{k=1}^{\infty} q_k(t) \sin \pi k x/L \qquad (12)$$

so that the Hamiltonian takes the form:

$$H = 1/2 \int_0^L (\dot{u}^2 + (\partial u/\partial x)^2) dx = L/4 \sum_k (\dot{q}_k^2 + (k\pi/L)^2 q_k^2)$$

(that is, the system is described as an infinite collection of harmonic oscillators each of "mass" L/2). The energy for the segment (0,a) is then given by this formula with the range of integration so restricted. For our purposes, it is enough to summarize the result: for the total square deviation ΔE^2 they found:

$$\Delta E^2 = (\Delta_1 E + \Delta_2 E)^2$$

where $\Delta_1 E$, $\Delta_2 E$ are defined by:

$$\Delta_1 E = 1/4 \sum_{j,k} q_j q_k a \left[f((j-k)\pi a/l) - f((j+k)\pi a/l) \right]$$

$$\Delta_2 E = 1/4 \sum_{j,k} q_j q_k a \left[f((j-k)\pi a/l) + f((j+k)\pi a/l) \right]$$

in which the summation with j=k is omitted, and f(x) is the function $f(x) = \frac{1}{x} \sin x$.

The mean square deviation is (by assumption) equal to the phase average; classically, in the limit in which L becomes large we may approximate the quantity

$\langle \Delta_1 E \, \Delta_2 E + \Delta_2 E \, \Delta_1 E \rangle$

by the integral

$$aL^2/8\pi \int_0^\infty \omega^2 \left[(q_\omega \dot{q}_\omega)^2 + (\dot{q}_\omega q_\omega)^2 \right] d\omega \qquad (13)$$

which vanishes by inspection. On the other hand:

$$\langle \Delta_1 E^2 + \Delta_2 E^2 \rangle = \langle E^2 \rangle / 2a \qquad (14)$$

as expected. What happens if we now consider the quantum theory? According to Born *et al* the result of Eq.(13), (14) still holds, and only two new factors come into play; first, the total mean energy <E> now includes the zero-point energy, and the mean square energy involves a cross-term between this and the mean energy (excluding the zero-point energy). It is this term which gives the first term on the RHS of Eq.(11) (the "particle" aspect). There also remains a term involving the square of the zero-point energy. But this is cancelled by the fact that Eq.(13) no longer vanishes, because of the commutation relation between the Fourier components:

$$\dot{q}_j q_k(n,n) - q_k \dot{q}_j(n,n) = -\frac{1}{2} \cdot \frac{2}{L} \cdot i\hbar \, \delta_{jk} \mathbb{I}(n,n) \qquad (15)$$

This is the only point at which Born *et al* explicitly define what the quantization of the classical string actually amounts to, and they introduce this equation with no justification other than the comment that "the quantity L/2 is to be regarded as the "mass" of the resonators". Presumably they have in mind the CCR:

$$p_j q_k(n,n) - q_k p_j(n,n) = -i\hbar \, \delta_{jk} \mathbb{I}(n,n)$$

with the identification $p_j = m\dot{q}_j = L\dot{q}_j/2$. In particular the authors did not identify the canonically conjugate variables from the Lagrangian theory of the classical string, and define the CCR's accordingly[6]. In retrospect, whilst this

[6] Note that there still remains a factor of 1/2 unaccounted for in Eq.(15) using the CCR between position and momentum; this is eliminated in the Lagrangian theory.

step was within their grasp (the authors acknowledge Dirac's paper of [1925], in which the canonical theory was formulated), the shortcomings of this early foray into QFT were the shortcomings of the matrix mechanics: the absence of the transformation theory, and more generally the representation theory of quantum observables.

Born *et al* made only one additional significant inroad on a QFT. They emphasised that in considering the expansion of Eq.(12) the coordinate x therein remains a classical quantity (a c-number in Dirac's terminology),

> since if in place of the continuous string we consider an elastic *series of points*, x would denote the *number* (multiplied by the lattice constant) of any given point (Born *et al* [1926].

1.1.3. Dirac and action-angle variables.

But QFT did not evolve in this straightforward way. From this point on the theory became entirely the creation of Dirac, and there is no doubt that he essentially constructed a full-blown (but non-relativistic) QED without the help of either the transformation theory, the distinction between operators and states, or for that matter the wave mechanics. In fact all these additional ideas found their way into the historic paper of 1927, (discussed shortly), but then this paper developed several routes to a QFT. One of these routes was provided by Dirac's systematic analysis of action-angle variables in quantum theory. In retrospect, it is clear that the fundamental formal technique which Dirac exploited contradicted the one clear (and correct) insight of the treatment of Born *et al*; that the spacetime coordinates entered as c-numbers into the theory. It is hard to imagine a more ironic course of events.

Action-angle variables pervade almost all of Dirac's work from his first paper of [1925] up to the beginnings of the electron theory. The relevant papers are:

A: The fundamental equations of quantum mechanics [1925].

B: Quantum mechanics and a preliminary investigation of the

hydrogen atom [1926a].

C: Relativity Quantum Mechanics with an application to Compton scattering [1926b].

D: The Quantum Algebra [1926c].

E: On the theory of Quantum Mechanics [1926d].

F: The Compton effect in Wave Mechanics [1926e][7].

In A Dirac introduced his terminology "q-numbers" (q for "queer" or "quantum"), and made some preliminary observations concerning the CCR's and matrix elements of combinations of the action-angle variables J_r, ω_r, in particular the functions:

$$\xi = J^{1/2}e^{i\omega}$$
$$\eta = -iJ^{1/2}e^{-i\omega} \qquad (16)$$

(for convenience I consider a system of 1 degree of freedom; these quantities are intimately related to the creation and annihilation operators). He noted in particular that one could recover the usual position and momentum variables[8], for the harmonic oscillator:

$$p = (\xi + i\eta)/\sqrt{2}$$
$$q = (i\xi + \eta)/\sqrt{2} \qquad (17)$$

whereupon the Hamiltonian takes the form

$H = \pi(q^2 + p^2) = 2\pi J.$

The Poisson bracket relationship between the action angle variables $[\omega, J]_{PB} = 1$ leads to the CCR:

$[\xi, \eta] = i\hbar$

whilst he considered their matrix elements by definition to be of the form $\xi(m,n) = 0$ except when $m = n+1$, and $\eta(m,n) =$

[7] Dirac's output in 1926 was phenomenal. To this list must be added "The physical interpretation of the quantum mechanics" [1926f] (on the transformation theory) and "On the elimination of nodes in quantum mechanics" [1926g] (a minor, and purely technical, paper which was once again concerned primarily with action-angle variables). There is also the small matter of completing and submitting his Ph.D thesis in the summer of the same year.

[8] For a massive oscillator of mass m and frequency ν the mechanical position and momentum variables are given respectively by $q/\sqrt{m\nu}$ and $\sqrt{m\nu}\,p$, and the Hamiltonian by νH, where q,p, and H are as above. We have replaced J and ω as used by Dirac by $J/2\pi$ and $2\pi\omega$, in line with his later notation. All commutators are unchanged.

0 except when m = n-1. As a consequence $\xi\eta(nn) = -iN\hbar$, where N is an integer; since classically

$$\xi\eta = -iJ$$

it follows that $J = N\hbar$, as expected. Using instead the (classically equivalent) expression

$$(1/2)(\xi\eta + \eta\xi) = -iJ$$

one obtains instead

$$J = (N+1/2)\hbar.$$

The reason for Dirac's interest in action-angle variables was made clear in B, where their connection with the Heisenberg paper of [1925] was pointed out. We must now consider a system of many degrees of freedom, with action-angle variables J_r, ω_r. We consider the Fourier expansion, for multiply periodic systems, of an arbitrary dynamical variable q:

$$q = \sum_{\mathbf{a}} C_{\mathbf{a}} e^{i\mathbf{a}.\mathbf{w}} = \sum_{\mathbf{a}} e^{i\mathbf{a}.\mathbf{w}} C'_{\mathbf{a}} \qquad (18)$$

where $\mathbf{w} = (\omega_1, \omega_2, \dots)$ and $\mathbf{a} = (a_1, a_2, \dots)$, $a_i \in \mathbb{Z}$. The C's, C''s are functions of the (q-numbers) J_r. The ω's are also q-numbers, and if the Hamiltonian is a function of the J's only it follows that their time derivatives are constants. Dirac showed that for any q-numbers p_r, q_r obeying the CCR's:

$$[q_r, p_s] = i\hbar\,\delta_{rs}$$

then it follows that

$$[e^{i\mathbf{a}.\mathbf{q}}, p_r] = i a_r e^{i\mathbf{a}.\mathbf{q}}$$

and for any function $f(q_r, p_r)$ of the q's and p's:

$$f(q_r, p_r) e^{i\mathbf{a}.\mathbf{q}} = e^{i\mathbf{a}.\mathbf{q}} f(q_r, p_r + a_r\hbar). \qquad (19)$$

Therefore when the Hamiltonian is a function only of the J's, $H = H(J_r)$, we have that

$$\begin{aligned} e^{i\mathbf{a}.\mathbf{w}} H(J_r) &= H(J_r - a_r\hbar) e^{i\mathbf{a}.\mathbf{w}} \\ H(J_r) e^{i\mathbf{a}.\mathbf{w}} &= e^{i\mathbf{a}.\mathbf{w}} H(J_r + a_r\hbar) \end{aligned} \qquad (20)$$

so that if we write:

$$\frac{d}{dt} e^{i\mathbf{a}.\mathbf{w}} = i\mathbf{a}.\dot{\mathbf{w}}\, e^{i\mathbf{a}.\mathbf{w}} = i e^{i\mathbf{a}.\mathbf{w}}\, \mathbf{a}.\dot{\mathbf{w}}'$$

we obtain the q-numbers $\mathbf{a}.\dot{\mathbf{w}}$, $\mathbf{a}.\dot{\mathbf{w}}'$ from the commutator:

$$\frac{d}{dt} e^{i\mathbf{a}.\mathbf{w}} = [e^{i\mathbf{a}.\mathbf{w}}, H]$$

and Eq.(19) as follows:

$$\begin{aligned} \hbar\, \mathbf{a}.\dot{\mathbf{w}} &= H(J_r) - H(J_r - a_r\hbar) \\ \hbar\, \mathbf{a}.\dot{\mathbf{w}} &= H(J_r + a_r\hbar) - H(J_r). \end{aligned} \qquad (21)$$

On the basis of Eq.(21) Dirac interpreted the $\mathbf{a}.\dot{\mathbf{w}}$, $\mathbf{a}.\dot{\mathbf{w}}'$ as q-number versions of the Bohr transition frequencies (which they would equal if Eq.(21) were a c-number equation). The ω_r themselves he tentatively identified as q-number versions of the Bohr orbital frequencies. There is a fascinating and, I suspect, profound, connection with the Heisenberg theory in its original form, now pointed out by Dirac. Let us suppose that, in a Fourier expansion of the form Eq.(18), the function $C_{\mathbf{a}}(\mathbf{J})$ is of the form $C(\mathbf{J},\mathbf{J}-a\hbar)$. As Dirac noted, the dependence of C on **a** and **J** is in some sense symmetric, for a real (*i.e.* self adjoint) magnitude x may be written:

$$x = \sum_{\mathbf{a}} C_{\mathbf{a}}(\mathbf{J})e^{i\mathbf{a}.\mathbf{w}} = x^* = \sum_{\mathbf{a}} e^{-i\mathbf{a}.\mathbf{w}}C_{\mathbf{a}}(\mathbf{J})^*$$

but from Eq.(19) the RHS may be written $\sum_{\mathbf{a}} C_{\mathbf{a}}(\mathbf{J} + \mathbf{a}\hbar)^* e^{-i\mathbf{a}.\mathbf{w}}$ so one has:

$$C_{-\mathbf{a}}(\mathbf{J}) = C_{\mathbf{a}}(\mathbf{J} + \mathbf{a}\hbar)^*.$$

In the new notation this equation takes the symmetric form:

$$C(\mathbf{J}, \mathbf{J}+\mathbf{a}\hbar) = C(\mathbf{J}+\mathbf{a}\hbar, \mathbf{J})^*.$$

Consider now the expansion of two variables x,y, and their product xy:

$$x = \sum_{\mathbf{a}} X(\mathbf{J}, \mathbf{J}-\mathbf{a}\hbar)e^{i\mathbf{a}.\mathbf{w}}$$

$$y = \sum_{\mathbf{b}} Y(\mathbf{J}, \mathbf{J}-\mathbf{b}\hbar)e^{i\mathbf{b}.\mathbf{w}}$$

$$xy = \sum_{\mathbf{c}} XY(\mathbf{J}, \mathbf{J}-\mathbf{c}\hbar)e^{i\mathbf{c}.\mathbf{w}}$$

It follows that

$$xy = \sum_{\mathbf{a},\mathbf{b}} X(\mathbf{J}, \mathbf{J}-\mathbf{a}\hbar)e^{i\mathbf{a}.\mathbf{w}}Y(\mathbf{J}, \mathbf{J}-\mathbf{b}\hbar)e^{i\mathbf{b}.\mathbf{w}}$$

which by Eq.(19) may be written:

$$xy = \sum_{\mathbf{a},\mathbf{b}} X(\mathbf{J}, \mathbf{J}-\mathbf{a}\hbar)Y(\mathbf{J}-\mathbf{a}\hbar, \mathbf{J}-\mathbf{b}\hbar-\mathbf{a}\hbar)e^{i(\mathbf{a}+\mathbf{b}).\mathbf{w}}$$

so that we conclude:

$$XY(\mathbf{J}, \mathbf{J}-\mathbf{c}\hbar) = \sum_{\mathbf{a}} X(\mathbf{J}, \mathbf{J}-\mathbf{a}\hbar)Y(\mathbf{J}-\mathbf{a}\hbar, \mathbf{J}-\mathbf{c}\hbar)$$

which is precisely a q-number version of the Heisenberg multiplication rule, Eq.(5).

This theory does not appear to have any straightforward interpretation; but I believe its remarkable elegance explains Dirac's interest in action-angle variables, and I suggest that if properly interpreted one will have a new insight into the meaning of the Heisenberg-Kramers dispersion theory and Heisenberg's point of entry into the matrix mechanics. It may also offer a new interpretation of the Thomas-Kuhn sum rule, the precursor of the CCR's;

however the difficulties of interpreting this formalism become clear, when we notice that making sense of the angle operators ω_r, is tantamount to the definition of a time operator. Classically ω_r is of the form $\phi_r - \nu_r t$, and the explicit solution for the position and momentum variables (*cf.* Eq.(17) and fn. 8) is:

$x(t) = 2A \cos(\phi - \nu t)$

$p(t) = -2m\nu A \sin(\phi - \nu t)$

where $A = (2m\nu)^{-1/2} J^{1/2}$. The time derivative of the ω's is precisely the frequency ν. Dirac fully appreciated this fact, and his papers on the Compton effect (C and F) were attempts to exploit the action-angle theory to provide a relativistic quantum mechanics in which the position and time coordinates were introduced as q-numbers[9].

They were also attempts to exploit the fundamental relationship, Eq.(19), most typically in the form of Eq.(20), to describe the process of the emission and absorption of radiation. The idea was to consider the radiation field as a perturbation of the form $H_I = A\cos(t-x/c).p$, where x refers to the particle position along the x-axis (aligned in the direction of the incident radiation), with t-x/c the (q-number) angle variable. This interaction was then expressed in the form:

$$H_I = A.\left[\, p(J)e^{i\omega} + e^{i\omega}p(J) \,\right] \qquad (22)$$

with the idea that its matrix elements defined the possible transitions of the atomic system, associated (heuristically) with photon emission and absorption, and (mathematically) with the change in action variables via Eq.(19) (note the importance in the ordering of the action and angle variables in Eq.(22) in this respect). Retrospectively, we may say that Dirac exploited certain properties of creation and annihilation operators to describe radiative transitions, but acting on the atomic system only and not

[9] The difficulties of making sense of a time-operator are well known, as also the inconsistency of CCR's between self-adjoint operators, one of which has a discrete spectrum. See Carruthers and Nieto [1968] for a readable review of time and angle operators in quantum theory.

the radiation field itself.

Dirac's analysis made no reference to the distinction between q-numbers and states; as a result much of his reasoning appears circuitous and imprecise. For example, a dominant heuristic was that pre-factors and post-factors in certain expressions refer to the atomic system prior to, and consequent upon, a radiative emission, but it is difficult to attach any precise meaning to this idea. In this he shared the limitation of the Göttingen approach, and like this group he was initially hostile to the wave mechanics (Weiner [1977 p.133]). One might think that in F these deficiencies would be removed, but in fact the wave mechanics is only used at one point in this paper (to remove an ambiguity in the ordering of the dynamical variables in the Hamiltonian), which otherwise follows the earlier treatment closely[10].

This question, of the ambiguity of the ordering in the transcription of c-number expressions, is absolutely central to all subsequent developments of algebraic methods in quantum theory. In Jordan's later abstract theory of quantum magnitudes, the difficulty was confronted *ab initio*, with remarkable success (see Sections 2.1, 2.3). Dirac, on the other hand, used piecemeal methods, despite his promising

[10] Jammer declares that "in the second article on the Compton effect..he virtually withdrew his former approach, calling it "rather artificial"" (Jammer [1974 p.141]) whilst Mehra records that "Dirac had worked on the problem for several years. He had come to a point where his method, that of (action-angle) variables which he had taken over from classical mechanics, seemed to fail. Right at that moment he came across Schrodinger's work which gave him the key to the solution" (Salam and Wigner [1972 p.37]). These comments are misleading; he did indeed refer to his earlier paper as "rather artificial", but he was referring to a certain method for writing down the Hamiltonian, and he immediately continued: "A more natural and more easily understood way of obtaining the matrices is provided by Schrodinger's wave mechanics" ([C,p.500]). See also the note added in proof: "...in (the present paper)...the wave equation is used merely as a mathematical help for the calculation of the matrix elements, which are then interpreted in accordance with the matrix mechanice."

attempt to devolop a general theory of q-number functions in D[11].

It is only with E that he begins to use analytic methods. This paper contained his first (and remarkably successful) foray into the wave mechanics: he showed that Bose (Fermi) statistics required that the wave function be symmetric (antisymmetric) under interchange of particle labels, developed the time-dependent perturbation theory, and also created the semi-classical treatment of radiation that is now standard in introductory texts (*e.g.* Schiff [1968]). Essentially this consisted of the application of this method to the Hamiltonian

$H = H_0 + \mathbf{A}.\mathbf{p}/c$

in which $\mathbf{A}$ is the vector potential, a c-number function of the particle coordinates and the time, and H_0 is the free Hamiltonian. To obtain an estimate on the total power radiated by an ensemble of atoms, he used the simple heuristic of taking the energy of a photon of frequency $\nu(n,m)$ associated with a transition from the state m to the state n, multiplied by the probability per unit time of its emission $|q(n,m|^2$, where q is the electric dipole moment, and then summing over all frequencies. This result differed by a factor of $\hbar\nu$ in comparison to Jordan's earlier treatment. This heuristic also led to some odd terminology, for example in expanding the wave function for the system (explicitly a one-particle system), $\psi(\mathbf{x},t) = \sum_n a_n(t)\psi_n(\mathbf{x})$, with $\{\psi_n\}$ an orthonormal basis in which the free Hamiltonian is diagonal, Dirac interpreted $a_n(t)\overline{a_n(t)}$ as the number of atoms in the nth energy state at time t. This may have been no more than a *façon de parler*, a way of quickly passing over to an ensemble interpretation of quantum probability, but it may equally indicate that at this time Dirac was in the habit of thinking of the wave-function as associated

[11] This paper actually addressed the question of defining functions of q-numbers which are not given as power series. He did not address the problem of ambiguity in transcribing c-number formulae, and made no further reference to this paper.

with the number of particles per unit volume. If so this gives a clue to the second quantization technique developed in Dirac [1927a]; on the other hand he was not at this point considering a quantization of the radiation field itself (nor was this idea explicitly discussed in any of his writing up to this time). Whilst in E he succeeded in obtaining the Einstein coefficients for the induced emission and stimulated absorption of radiation, he failed to obtain the coefficient for spontaneous emission. On this point he remarked:

> One cannot take spontaneous emission into account without a more elaborate theory involving the positions of the various atoms and the interference of their individual emissions, as the effects will depend upon whether the atoms are distributed at random, or arranged in a crystal lattice, or all confined in a volume small compared with a wave-length...[E p.677]

That is, he did not regard this failing as a consequence of having failed to treat the dynamics of the radiation field itself[12]. Indeed in C , in connection with the formula for the intensity of the incident radiation $I = cA^2\nu^2/8\pi$, he argued: "since ν and I can be measured physically they are c-numbers, and therefore so is **A**" [C p.415].

The remarkable paper of 1927 was a synthesis of the methods and results of E with those of his papers on the Compton effect; that is, the combination of the wave mechanics, with its attendant notion of state, the symmetrization condition, to express Bose statistics, and the perturbation theory (all these in E), with the method for describing radiative transitions through the action angle variables. In fact this last, combined with the perturbation theory, was sufficient

[12] The success of the QFT, developed shortly afterwards, in accounting for spontaneous emission (but not in the way indicated above!) was particularly emphasised by Dirac as evidence for its generality. However, and despite the simple heuristic that the spontaneous emission is actually an induced emission (induced by the zero-point fluctuations), the induced coefficient can be obtained by a simple elaboration of the semi-classical theory. See Schiff [1968 p.408-14]; in retrospect Born and Jordan [1925] were a short step to deriving the A and B coefficients in this way.

in itself to generate a full QED including the reaction on the radiation field (what I shall call his method 1). Together with the notion of state and the symmetrization condition, enough to generate a free QFT (method 2); and together with the matter-wave heuristic of the wave-mechanics, combined with a straightforward canonical quantization, enough to generate the same free QFT (method 3). This last method he considered purely formal, a shortcut to his method 2; the equivalence of method 2 and 3 but established this fact. The equivalence of method 2 with method 1 was, on the other hand, fundamental: the consistency of the light-quantum and field was thus established. It was left to Jordan to point out that his method 3 constituted a genuine and independent field quantization. The existence and equivalence of methods 2 and 3 were then established by Jordan, Klein and Wigner for a wide class of interactions in the non-relativistic theory, and the method generalized to deal with fermions also.

These ideas are now elaborated in detail; they constitute the subject matter of the following two sections. However before commencing their study there is one further route to a quantum field theory which cannot be omitted, although it had little or no direct influence on Dirac or the tradition within which he worked[13]. That is the theory of quantum statistics applied to *material* particles. On the basis of this theory, one might have been led directly to Dirac's method 3, and anticipated its equivalence to method 2.

[13] It is just for this reason that he did not consider his method 3 as anything more than a mnemonic. I know of only one commentary explicited devoted to Dirac's paper of [1927a] (Bromberg [1979]); she on the contrary considers the Einstein gas theory the primary influence leading to this paper, and makes no reference to his earilier work on Compton scattering. Neither does she distinguish methods 2 and 3.

1.1.4. Quantum statistics and massive particles.

From the work of historians like Martin Klein [1964] and Linda Wessels [1979] the complex web of relationships between the investigation of quantum statistical mechanics, by the likes of Debye, Einstein, Planck and Schrödinger, and the de Broglie theory of matter waves, has been carefully mapped out. In the development of quantum statistical mechanics one sees the slow and painful shift away from the Boltzmann statistics and towards a statistics of indistinguishable particles; the difficulties lay in the extensivity of entropy, and the apparent necessity of including a factor of 1/n! in the partition function (Sackur [1911], Tetrode [1912], in order to obtain additivity. This issue had been the subject of a heated controversy between Planck and Ehrenfest[14]; now too Schrödinger entered the debate. With the Einstein gas theory, both Planck and Schrödinger (suspicious of the new statistics) were led to consider other alternatives; in particular Schrödinger was led to the de Broglie theory of statistics in application to material particles, rather than photons (or "photon aggregates", as de Broglie considered them). This work led directly to the wave mechanics.

Klein and Wessels consider the influences upon Schrödinger, and his route to the wave mechanics, not the development of QFT. Whilst Schrödinger was led to de Broglie's work in the context of statistical mechanics, his statistical ideas had tenuous connections with the 1-particle wave mechanics. On the contrary, they have profound connections with quantum field theory.

Early conceptions of QFT arose purely from statistics, and it was Einstein who had laid down the fundamental heuristics of the quantized radiation field. Following the

[14] Planck [1921]; Ehrenfest and Trkal [1921]. The controversy over quantum statistics now enters its third decade.

Bose paper of [1924], he realized that from a statistical point of view, the wave-like properties of radiation could be traced to the new statistics. He then considered the consequences of assuming the new statistics for a *particle* ensemble:

> The interest of the theory lies in the fact that it is based on the hypothesis of a far-reaching formal relationship between radiation and gas. According to this theory, the degenerate gas deviates from the gas of (ordinary) statistical mechanics in a way analogous to that in which the behaviour of radiation, according to Planck's law, deviates from its behaviour according to Wien's law. If Bose's derivation of the Planck radiation formula is to be taken seriously, then one may not also pass up this theory of the ideal gas; for if one is justified in considering radiation as a gas of quanta, then the analogy between the gas of quanta and the gas of molecules must be complete. (Einstein [1925 p.3]).

In this same paper he considered the analysis of fluctuation in particle number, the analogue of his earlier analysis of fluctuations in energy for the radiation field, that was so influential in defining the problem of the wave-particle dualism. He found precisely the same dualism present: a term that was typical of a particle ensemble, and a term typical of a *field* theory. The implication is that this latter can be explained

> ...if one associates a radiation process with the gas in a suitable way, and calculates its interference fluctuation. I go into this interpretation in more detail because that I believe that there is more than a mere analogy involved here. In a very noteworthy work, de Broglie has shown how one can associate a (scalar) wave field with a material particle or a system of material particles. (Einstein [1925 p.9]).

This reference, bearing the extraordinary authority of Einstein, brought de Broglie's work to the attention of every theoretician working in quantum statistics, amongst them Planck and Schrödinger. Neither of them were happy with the new statistics; both took up one of de Broglie's central discoveries, that one can establish the Planck distribution on the basis of *Maxwell-Boltzmann* statistics if only one considers all photons of the same frequency as a "quantum aggregate", a normal mode of the radiation field "containing" all of these photons. The Maxwell-Boltzmann

statistics applied to the quantum aggregates immediately led to the Planck distribution. De Broglie explained the failure of Maxwell-Boltzmann statistics applied to the individual quanta (photons, despite his terminology) as follows:

> If two or more atoms have phase waves that exactly superpose, by which one may say therefore that they are transported by the same wave, their movements may no longer be treated as entirely independent and these atoms may no longer be treated as distinct entities in the calculation of probability. (de Broglie [1925 Ch.7. Sec.3])

What, however, was to count as a "wave" for a material gas? First Planck, and then Schrödinger, tried to get away from the customary practise of applying the quantum conditions to the individual molecules of the gas, and instead to the gas as a whole; the partition function derived in this way could still be associated with

> an individual atom, though not in empty but in gas-filled space of volume V because the count of summed terms is not given through quantization of a single atom in volume V, but through the previously performed quantization of the states of the gas (Planck [1925a p.50]).

Schrödinger was increasingly drawn to this idea:

> The energy levels...of the gas molecules must now, of course, be derived from the energy level distribution of the body of gas as a whole, exactly the opposite of how it was previously done (Schrödinger [1925 p.439])

In the last of Schrödinger's papers on quantum statistics, "On Einstein's Gas Theory", completed in mid-December of 1925 and immediately preceding the series of papers introducing the wave mechanics, he summarized his conclusions as follows:

> Einstein's theory of a gas is obtained by applying to the gas molecules that form of statistics that leads to the Planck radiation law when it is applied to "atoms of light". However, one can also obtain the Planck radiation law by using "natural statistics", if one applies them to so-called "aether oscillators", that is to the degrees of freedom of the radiation. The light atoms then appear only as the energy levels of the aether oscillators...one must therefore simply form a picture of the gas like the picture of cavity radiation that does not correspond to the extreme light-quantum representation: the natural statistics ...will then lead to Einstein's gas theory (Schrödinger [1926a p.95]).

The circle of ideas has been completed: the radiation field, on empirical grounds, displays (Maxwell-Boltzmann) probabilistic behaviour typical of both waves and particles; the normal modes must be quantized, and the excitation number gives the number of photons. It may be considered wholly particulate only with the assumption of Bose statistics. The assumption of Bose statistics for *material* particles resolves long-standing difficulties in the definition of an extensive entropy; favouring Maxwell-Boltzmann statistics instead, and yet to obtain an additive entropy, we must consider the material gas as a field also, and once again the molecules must correspond to the excitations in the normal modes of this field.

It is remarkable that one is led to this conception of a material QFT through the *rejection* of Bose statistics.

Section 1.2. Quantum Electrodynamics and the Field - Many-particle Equivalence.

> Yet the postulate that lies at the root of every scientific enquiry, the act of faith which has always sustained scientists in their unwearying search for explanation, consists in the assertion that it must be possible - although perhaps at the heavy cost of ideas held for long and concepts of proved usefulness - to reach a synthetic view uniting all the partial theories suggested by the various groups of phenomena, and embracing them all despite their apparent contradictions.
>
> L. de Broglie

1.2.1. The quantum theory of the emission and absorption of radiation: organization.

We have elaborated the theoretical background to this paper; it must be emphasised that Dirac, along with Jordan, was also concerned to free the new mechanics from any dependence on the correspondence principle. In the introduction to his second paper on the new QED, in which (amongst other things) he derived the Heisenberg-Kramers dispersion theory, it was the principal result cited:

> The new quantum mechanics could at first be used to answer questions concerning radiation only through analogies with the classical theory. In Heisenberg's original matrix theory, for example, it is assumed that the matrix elements of the polarization of an atom determine the emission and absorption of radiation analogously to the Fourier components in the classical theory. In more recent theories (i.e. the wave mechanics) a certain expression for the electric density obtained from the quantum mechanics is used to determine the emitted radiation by the same formulae as in the classical theory. These methods give satisfactory results in many cases, but cannot even be applied to problems where the classical analogies are obscure or non-existent, such as resonance radiation and the breadths of spectral lines.
>
> A theory of radiation has been given by the author which

> rests on a more definite basis. It appears that one can treat a field of radiation as a dynamical system, whose interaction with an ordinary atomic system may be described by a Hamiltonian function. (Dirac [1927b p.710]).

That said, it is doubtful that this concern played much rôle in motivating the initial breakthroughs of the [1927a] paper; in any case, he made no mention of it there. I believe it is more accurate to say that this concern underlay his interest in the action-angle variables and radiative interactions, but that he pursued this theory also because he wished to understand the remarkable connections with the original Heisenberg paper and the Bohr frequency conditions.

Consider now the q-number theory of action-angle variables. Cleary the angle variable defines a q-number expression of the time co-ordinate. In this connection Dirac does three things:

(i) takes over the action-angle formalism considered in (1.1.3).

(ii) explicitly declares that the angle coordinate satisfies CCR's with the Hamiltonian, and yet

(iii) explicitly denies that the time is treated as a q-number; in fact declares that:

> The theory is non-relativistic only on account of the time being considered as a c-number, instead of being treated symmetrically with the space-co-ordinates.

We resolve the puzzle with the observation that although Dirac works primarily in the Heisenberg picture, when he actually calculates the Einstein A and B coefficients he switches to the interaction representation, with an explicit c-number time dependence of the states. Further for most of this paper one can assume that the Hamiltonian has no explicit time-dependence, so that a q-number time does not directly intrude except in the definition of the creation and annihilation operators. On the other hand, with respect to method 1 the interaction Hamiltonian has an explicit time-dependence; making the time here a q-number (or classically a canonical variable) removes the explicit

time-dependence and makes the total Hamiltonian independent of time, and permits an ill-defined correspondence between this Hamiltonian and that constructed in methods 2 and 3. For the latter methods one can avoid the problem; since his method 1 is anyway ill-defined (and probably meaningless) I shall not pursue this puzzle further.

To convey Dirac's own interpretation of his results, we quote in full from §1, Introduction and summary:

> The underlying ideas of the theory are very simple. Consider an atom interacting with a field of radiation, which we may suppose for definiteness to be confined in an enclosure so as to have only a discrete set of degrees of freedom. Resolving the radiation into its Fourier components, we can consider the energy and phase of each of the components to be dynamical variables describing the radiation field. Thus if E_r is the energy of a component labelled r and θ_r is the corresponding phase (defined as the time since the wave was in a standard phase), we can suppose each E_r and θ_r to form a pair of canonically conjugate variables. In the absence of any interaction between the field and the atom, the whole system of field plus atom will be describable by the Hamiltonian
>
> $$H = \sum_r E_r + H_o \qquad (1)$$
>
> equal to the total energy, H_o being the Hamiltonian for the atom alone, since the variables E_r, θ_r obviously satisfy their canonical equations of motion
>
> $$\dot{E}_r = -\partial H/\partial\theta_r = 0, \quad \dot{\theta}_r = \partial H/\partial E_r = 1. \qquad (2)$$
>
> When there is an interaction between the field and the atom, it could be taken into account on the classical theory by the addition of an interaction term to the Hamiltonian Eq.(1), which would be a function of the variables of the atom and of the variables E_r, θ_r that describe the field. This interaction term would give the effect of the radiation on the atom, and also the reaction of the atom on the radiation field.
>
> In order that an analogous method may be used on the quantum theory, it is necessary to assume that the variables E_r, θ_r are q-numbers satisfying the standard quantum conditions
>
> $$\theta_r E_r - E_r \theta_r = i\hbar \qquad (3)$$
>
> like the other dynamical variables of the problem. This assumption immediately gives light-quantum properties to the radiation[1]. For if ν_r is the frequency of the

[1] Dirac here includes the footnote: "Similar assumptions have been used by Born and Jordan ([1925]) for the purposes of taking over the classical formula for the emission of

component r, $2\pi\theta_r$ is an angle variable, so that its canonical conjugate $E_r/2\pi\nu_r$ can only assume a discrete set of values differing by multiples of $\hbar$, which means that E_r can change only by integral multiples of the quantum $h\nu_r$. (Dirac [1927a p.244-245]; henceforward all references to this paper will be of the form [244-245]).

The organization of the paper is as follows:

§2. *The Perturbation of an Assembly of Independent Systems.*
The second quantization process is formulated. The 1-particle Schrödinger theory is replaced by a q-number formalism; Dirac interpreted this formalism *ab intio as a* formulation of the quantum mechanical description of an ensemble of systems which are independent and all subject to the same perturbation.

§3. *The Perturbation of an Assembly satisfying the Einstein-Bose Statistics.*
In this section he shows that the Hamiltonian constructed in §2 describes the effect of a perturbation on a system of an (undetermined number of) bosons. In a later terminology he invents the *occupation number representation.*

§4. *The Reaction of the Assembly on the Perturbing System.*
The Hamiltonian of §2 is written in a form which explicitly exhibits the variables of the atomic system, and those of the assembly of bosons.

§5. *Theory of transitions in a system from one state to others of the same energy.*
The Born scattering theory (the "Golden Rule No.2") is developed, with minor modifications, and essentially no reference to the foregoing.

§6. *Application to light-quanta.*
The Hamiltonian defined in §4, for an arbitrary interaction term between the atomic system and the bosons (for the first time explicitly identified as photons), is interpreted so as to distinguish transitions between states of the photons which have non-zero energy, and transitions to and from

radiation by a dipole into the quantum theory, and by Born et al ([1926]) for calculating the energy fluctuations in a field of black-body radiation". As we have seen, they did nothing of the kind.

a state in which the photon has zero energy. Photons which are described by this hypothetical state of zero energy are considered physically unobservable, so that the transitions in question are equivalent to single creation and annihilation events. Processes of the latter kind cannot be described by the Hamiltonians so far constructed except by this artifice.

§7. *The probability coefficients for emission and absorption.*

This section is entirely self-contained and makes no reference to the previous results; it is a direct continuation of Dirac's earlier work on the Compton effect, and concludes with a derivation of the Einstein A and B coefficients by means of the scattering theory of §5. Concerning the earlier material, he observes that no transitions between states of the same particle (photon) number can be described, but that for transitions in which single photons are created and destroyed the Hamiltonian is of the same form as that part of the Hamiltonian defined in §6, which describes transitions to and from the zero-energy state. On this basis he concludes that the wave point of view (as developed in this section) is consistent with the light-quantum point of view (developed in § 2,3,4,6).

In terms of the summary at the close of (1.1.3), method 1 is defined in §6, method 2 in §3, and method 3 in §2. I shall first discuss method 1.

1.2.2. Method 1: QED.

Dirac began with the Hamiltonian:

$H = H_o + H_F + H_I$

in which H_o is the free Hamiltonian for a 1-particle system, an electric dipole, H_F the free Hamiltonian for the radiation field confined to a finite volume V, and H_I the interaction Hamiltonian in the Coulomb gauge classically given by:

$$\mathbf{A}.\dot{\mathbf{x}}/e = \frac{1}{e}\sum_r A_r \dot{x}_r \quad . \qquad (4)$$

Here x_r is "the component of the total polarization of the atom in the direction of A_r, which is the direction of the electric vector of the component r" [262]. This requires some explanation; A_r is properly a vector quantity $\mathbf{A}_r$, the r^{th} term in the Fourier expansion of $\mathbf{A}$:

$$\mathbf{A} = \sum_r \mathbf{A}_r = \sum_r \mathbf{a}_r \cos 2\pi\nu_r(\phi_r - t)$$

so we would do better to distinguish the three spatial coordinates and write Eq.(5) as:

$$H_I = \frac{1}{e}\sum_r \mathbf{A}_r.\dot{\mathbf{x}}$$

or as:

$$H_I = \frac{1}{e}\sum_r (A_r^x \dot{x} + A_r^y \dot{y} + A_r^z \dot{z})$$

However to avoid complications we retain Dirac's original notation, but we emphasise that no Fourier expansion of the electric dipole moment is involved in Eq.(4).

Dirac defined the angle variable θ_r $(= \phi_r - t)$ in the passage quoted above; he now defines the new angle variables

$$\Theta_r = h\nu_r \theta_r$$

so that its canonical conjugate is just the number operator N_r with:

$$A_r = a_r \cos \Theta_r/\hbar.$$

Dirac now considers the a_r q-numbers (functions of the action variables) and determines their expression by the following argument: on the basis of the light-quantum heuristic it is clear (*cf.* the discussion at the close of (1.1.3)) that the intensity of radiation per unit frequency range about the frequency r is:

$$I_r = n_r h\nu_r^3/c^2 \qquad (5)$$

where n_r is the number of photons present of frequency ν_r

(the eigenvalues of the N_r). On the other hand, from the classical theory the flow of energy per unit area per unit time for the component r is $\pi a_r^2 \nu_r^2/2c$, so that I_r is also given by:

$$I_r = \pi a_r^2 \nu_r^2 \sigma_r /2c$$

where σ_r is the number of stationary waves in the finite volume V (of given polarization) in the interval $\{\nu_r, \nu_r + d\nu_r\}$[2]. Equating the two yields:

$$a_r = 2(\hbar\nu_r/\sigma_r c)^{1/2} n_r^{1/2}.$$

This result suggests the quantum mechanical expression for the q-number a_r:

$$a_r = 2(\hbar\nu_r/\sigma_r c)^{1/2} N_r^{1/2}.$$

In what follows we shall continue to indicate the eigenvalues of the number operator by n_r.

Dirac had now obtained a q-number expression for H_I; for H_F he assumed the expression:

$$H_F = \sum_r h\nu_r N_r$$

to obtain the total Hamiltonian:

$$H = H_o + \sum_r h\nu_r N_r + 2c^{-1}\sum_r (\hbar\nu_r/c\sigma_r)^{1/2} \dot{x}_r N_r^{1/2} \cos \Theta_r/\hbar \qquad (6)$$

The critical step now comes in the replacement of the term $N_r^{1/2}\cos \Theta_r/\hbar$ by the expression:

$$\frac{1}{2}\left(N_r^{1/2} e^{i\Theta r/\hbar} + e^{-i\Theta r/\hbar} N_r^{1/2}\right).$$

Dirac could offer no justification for the particular choice of the order of q-numbers in this expression. Using Eq.(19) of (1.1.3) he was now able to write the Hamiltonian in the form:

$$H = H_o + \sum_r h\nu_r N_r + \\ + 2c^{-1}\sum_r (\hbar\nu_r/c\sigma_r)^{1/2} \dot{x}_r \left[N_r^{1/2} e^{i\Theta r/\hbar} + (N_r+1) e^{-i\Theta r/\hbar}\right] \qquad (7)$$

which he interpreted as follows:

> The probability of a transition in which a light-quantum in the state r is absorbed is proportional to the square of the modulus of that matrix element of the Hamiltonian which refers to this transition. This matrix element must come from the term $N_r^{1/2} e^{i\Theta r/\hbar}$ in the Hamiltonian, and must therefore be proportional to n_r, where n_r is the

[2] Explicitly, $\sigma_r = V\nu_r^2 d\nu_r d\Omega/c^3$, where Ω is the measure for solid angle.

number of light-quanta in the state r before the process. In the same way the probability of a light quantum in state r being emitted is proportional to (n_r+1). [261].

With this Einstein's laws follow immediately[3]; for by Eq.(5)

$$n_r = (c^2/h\nu_r^3)\, I_r$$

so that

$$n_r + 1 = (c^2/h\nu_r^3)(I_r + h\nu_r^3/c^2)$$

and the ratio, which by the foregoing is the ratio of the probability of absorption to induced emission, is just

$$I_r/(I_r + h\nu_r^3/c^2)$$

in accordance with Einstein's laws (neglecting polarisation).

Dirac was also able, by application of the scattering theory developed in § 5, to actually calculate the probabilities for absorption and emission process, and obtained agreement with the Einstein results. We shall not go into the details here. Note that the spontaneous emission process is subsumed under the induced emission, since the latter is proportional to n_r+1.

Inspection of Eq.(7) together with the general heuristic, that the modulus square of its matrix elements are proportional to the probability of the transition between the associated states, shows that transitions are only possible between states which differ in total photon number by unity, and only when the operator $\dot{x}_r$ has non-vanishing

[3] In fact Dirac was in error in his identification; it is not the term $\sqrt{N}e^{i\Theta}$ which induces an absorption process, but the term $\sqrt{N+1}e^{-i\Theta}$, as follows from the relationship

$$e^{-i\Theta}N = (N+1)e^{-i\Theta}.$$

i.e. Dirac confused the creation with the annihilation operators. Nevertheless Dirac's conclusion (that absorption processes are proportional to $\sqrt{j}$) still follows, as is evident from the elementary computation (here j, j' refers to the stationary states of the atomic system):

$$\langle n-1, j \mid \dot{x}\sqrt{(N+1)}\, e^{-i\Theta} \mid n, j' \rangle = \langle n-1, j \mid \dot{x}e^{-i\Theta}\sqrt{N} \mid n, j' \rangle$$

$$= \sqrt{n}\, \langle n-1, j \mid \dot{x}e^{-i\Theta} \mid n, j' \rangle.$$

Dirac is in error of confusing the action of an operator on eigenvalues with the action on the states; this problem will arise again in (1.2.3).

matrix elements with respect to the states of the atomic system. Since the time derivative of x_r is defined with respect to the full Hamiltonian, the total energy must be conserved. The theory is not of course an interacting field theory, but a theory of the interaction of a 1-particle system, described according to the quantum mechanics, with a QFT, such that the 1-particle system is subject to perturbations only when the quantum field emits or absorbs a single photon (this is to zero order in perturbation theory; multiple emission and absorption processes will occur at higher orders). This should be compared to the theory of the Compton effect, in which exactly the same technique (writing the perturbing field in action-angle variables) was used to describe transitions in the variables describing the *atomic* system. In a sense Dirac simply switched the action of the action-angle variables from the states of the atomic system to the states of the field, whilst correlating any such transition with a perturbation acting on the states of the atomic system through the q-number $\dot{x}$.

1.2.3. Method 2: quantum mechanics of a boson ensemble

In this sub-section we shall slightly sharpen Dirac's treatment, which can in fact be made rigorous. Dirac essentially invents the occupation number representation, and expresses the Hamiltonian which results in terms of the number and phase operators. We need not introduce the phase operators at all, however.

Throughout we suppose that $\mathfrak{h}(i)$ is the 1-particle Hilbert space of (time-independent) states for particle i, (the same Hilbert space of each i), which is one of n distinguishable particles, and that $\mathcal{H} = \otimes_{i=1}^{n} \mathfrak{h}(i)$. The symmetrized subspace of $\mathcal{H}$, denote $\mathcal{H}_s$, will then be appropriate to the description of an ensemble of n *indistinguishable* particles[4]. The inner product on $\mathcal{H}$ is

[4]One way of formulating this idea is that we have an

induced by the inner product $\langle .,. \rangle$ on $\mathfrak{h}$ and is denoted $(.,.)$. We shall systematically translate Dirac's formula into this framework.

Dirac began with the 1-particle Schrödinger equation and the basis $\{u_j\}$ of $\mathfrak{h}(1)$ in which the free Hamiltonian is diagonal (we may assume the Hamiltonian has a discrete spectrum for convenience). The (time-independent) 1-particle Hamiltonian H is considered the sum of a free and interaction part:

$H = F + V$

The Schrödinger equation:

$H\psi(t) = i\hbar\partial\psi/\partial t$

may be written:

$$i\hbar \dot{b}_k = \sum_l \langle u_k, Hu_l \rangle b_l \qquad (8)$$

where

$$Fu_k = E_k u_k, \quad \psi(t) = \sum_k b_k(t) u_k$$

The u_k's are of course independent of time. We may also write Eq. (8) in the form:

$$i\hbar \frac{\partial}{\partial t} \langle u_k, \psi(t) \rangle = \langle u_k, H\psi(t) \rangle.$$

Dirac now[5] considered the problem of n independent systems all subject to the above evolution; indicating by H(i) the Hamiltonian of the i^{th} particle, and by the numbers $r_1, r_2, \ldots, r_n$ the stationary states of the assembly of systems, he concluded that the Schrödinger equation becomes:

$$i\hbar \dot{b}(r_1 r_2 \ldots) = \sum_{s_1, s_2, \ldots} H_A(r_1 r_2 \ldots; s_1 s_2 \ldots) b(s_1 s_2 \ldots) \qquad (9)$$

where $H_A = \sum_i H(i)$, and $H_A(r_1 r_2 \ldots; s_1 s_2 \ldots)$ is the matrix element which vanishes when more than one s_i differs from the corresponding r_i; equals $H_{r_i s_i}$ when s_i differs from r_i and every other s_k equals r_k; and equals $\sum_i H_{r\, r}$ when every

ensemble of indistinguishable particles which are "transcendentally individuated" by a naming operation, which we need not consider can be physically implemented. This naming operation leads to a redundancy in the description which is afterwards removed by restriction to the symmetric subspace. For details see French [1984], Redhead [1983].

[5] Dirac was not specific on this point, and on occasions referred to the number as "undetermined". The reason will become clear shortly.

s_1 equals r_1.

We can formulate this result in a more transparent way as follows; define H_A as:

$$H_A = H(1)\otimes\mathbb{I}\otimes\ldots\otimes\mathbb{I}\otimes\ldots \oplus \mathbb{I}\otimes H(2)\otimes\mathbb{I}\otimes\ldots\oplus\ldots\oplus \mathbb{I}\otimes\ldots\otimes H(n) \qquad (10)$$

From Eq.(10) it is clear that indeed:

$$(u_{r_1}u_{r_2}\ldots u_{r_i}\ldots,\ H_A u_{s_1}u_{s_2}\ldots u_{s_i}\ldots)$$

$$= \begin{cases} 0 \text{ if there exists } i,j \text{ such that } i\neq j \text{ and } r_i\neq s_i,\ r_j\neq s_j \\ \langle u_{r_j}, H(j)u_{s_j}\rangle \text{ if } r_j\neq s_j \text{ and for all } k\neq j,\ r_k = s_k. \\ \sum_i \langle u_{r_i}, H(i)u_{s_i}\rangle \text{ if for all } i\ r_i = s_i \end{cases} \qquad (11)$$

where $u_{r_1}u_{r_2}\ldots = u_{r_1}\otimes u_{r_2}\otimes\ldots \in \mathcal{H}$ (to save on notation, we denote such states simply $u_{r_1r_2\ldots}$). The Hamiltonian H_A clearly has the interpretation, that it *either* leaves the state of the assembly unchanged *or* that it induces a transition in the state of *one and only one particle.* This is obvious by inspection of Eq.(10); we should not wonder at this restriction (for there are many other Hamiltonians *a priori* possible to describe the dynamics of a particle ensemble); this is explicitly an evolution *induced* by a 1-particle evolution, in a way which is made precise by the foregoing, and which defines an evolution which is the *canonical second quantization* of a 1-particle evolution.

Clearly an arbitrary state $\psi(t) \in \mathcal{H}$ (for each value of t) can be expanded:

$$\psi = \sum_{r_1r_2\ldots} b_{r_1r_2\ldots}(t)u_{r_1r_2\ldots}$$

so that Eq.(9) is obtained, and by means of Eq.(11):

$$i\hbar\dot{b}_{r_1r_2\ldots r_i\ldots} = \sum_k \sum_{s_k\neq r_k} \langle u_{r_k}, H(k)u_{s_k}\rangle\, b_{r_1r_2\ldots r_{k-1}s_k r_{k+1}\ldots} + \sum_j \langle u_{r_j}, H(j)u_{r_j}\rangle\, b_{r_1r_2\ldots}(t). \qquad (12)$$

Dirac now selects those states which are completely symmetric under interchange of particle labels, that is such that a state of the form

$$\psi = \sum_{r_1r_2\ldots} C_{r_1r_2\ldots} u_{r_1}\otimes u_{r_2}\otimes\ldots$$

satisfies:

$$\sum_{r_1 r_2 \ldots} C_{r_1 r_2 \ldots} u_{r_1} \otimes u_{r_2} \otimes \ldots = \sum_{r_1 r_2 \ldots} C_{r_2 r_1 \ldots} u_{r_2} \otimes u_{r_1} \otimes \ldots$$

which is ensured if the complex numbers $C_{r_1 r_2 \ldots}$ are completely symmetric in the $r_1 r_2 \ldots$. For such a state, Dirac realized it is sufficient to specify *how many* particles are in *which* state - that is, to specify a string of positive integers $n_1, n_2, \ldots n_r \ldots$ where the index set $\{1,2,..r,..\}$ *labels the orthonormal basis* $\{u_r\}$. Previously, each r_i took on values in this index set; the index i, however, labelled the particles.

If we suppose the states $u_{n_1 n_2 \ldots} \in \mathcal{H}_S$ form an orthonormal basis in $\mathcal{H}_S$ then expanding an arbitrary state $\psi(t) \in \mathcal{H}_S$ in terms of this basis will lead to new expansion coefficients $b_{n_1 n_2 \ldots}(t)$; in order to preserve the normalization of the state it is necessary that:

$$\sum_{r_1 r_2 \ldots} |b_{r_1 r_2 \ldots}(t)|^2 = 1 = \sum_{n_1 n_2 \ldots} |b_{n_1 n_2 \ldots}(t)|^2 \qquad (13)$$

(which follows on the assumption that $\{u_{n_1 n_2 \ldots}\}$ is a normalized set of basis vectors). We cannot just identify a given completely symmetric $b_{r_1 r_2 \ldots}$ (regarded as a function of the r_i's) with that function of the n_r's, which equals $b_{r_1 r_2 \ldots}$ when n_1 of the r_i's take the value 1, n_2 of the r_i's take the value 2, and so on, because there are many ways in which these occupation numbers may be specified by a set of integers $r_1, r_2 .. r_i \ldots$, and b (as a function over all such sets) has been normalized to take account of this (by assumption it is normalized and completely symmetric).

This solution to this problem was well-known; there are precisely $n!/n_1!n_2!\ldots n_r!$ such sets of integers $r_1, r_2, .. r_i \ldots$ with the same occupation numbers $n_1, n_2, .., n_r, ..$ summing to n (the total particle number) and therefore we must have:

$$b_{n_1 n_2 \ldots n_r \ldots} = (n!/n_1!n_2!\ldots n_r!..)^{1/2} b_{r_1 r_2 .. r_i ..} \qquad (14)$$

We now write the Hamiltonian of Eq.(12) in this representation using Eq.(14); we need only note that the matrix element $\langle u_{r_i}, H(i) u_{s_i} \rangle$ is that of the 1-particle

Hamiltonian (identical for each particle) between the states defined by the numbers r_i, s_i. Since it does not matter which particle is involved in this transition, we may denote these states simply by r,s, noting that it is multiplied by the coefficient $b_{r_1 r_2 \ldots r_{i-1} s_i r_{i+1} \ldots}$ - that is to say when the i^{th} particle in the state r_i is replaced by a particle in the state s_i - so that it should now be multiplied by the coefficient $b_{n_1 \ldots n_r - 1, \ldots, n_s + 1, \ldots}$, that is to say in which there is one less particle in the state r, and one more in the state s[6].

The result (after removal of an overall factor $(n!/n_1! \ldots)^{1/2}$) is the equation:

$$i\hbar \dot{b}_{n_1 n_2 \ldots}(t) = \sum_r \sum_s n_r^{1/2} (n_s + 1 - \delta_{rs})^{1/2} \langle u_r, H u_s \rangle b_{n_1 n_2 \ldots n_r - 1, \ldots, n_s + 1 \ldots}(t) \qquad (15)$$

(when s=r the last factor is just $b_{n_1 n_2 \ldots n_r \ldots n_s \ldots}(t)$).

So far we have simply the quantum mechanics of a system of n bosons. With two critical steps we have instead a quantum field theory. The first step is to express the occupation numbers in terms of (the action of) number operators; once this is done the resulting Hamiltonian is independent of the precise number of particles present. The second step is then to construct a Hilbert space which also is independent of the number of particles present, in the sense that the total particle number becomes one more piece of information, along

[6] we warn the reader that this heuristic is misleading in the important respect, that this does not mean that this term in the summation of Eq.(12) is associated with the transition of the ith particle from the state r_i to the state s_i, but rather from the state s_i to the state r_i. It is the growth of the expansion coefficient $b_{r_1 r_2 \ldots r_m \ldots}$ that is controlled by this equation, and we see that it increases proportional to the expansion coefficient $b_{r_1 r_2 \ldots r_{m-1}, r_s, r_{m+1}, \ldots}$, that is to say according to the extent to which the system is found to have the sth particle in the state r_s. The transition in question is from the state r_s to the state r_m.

with the occupation numbers for the various states, to be specified by the state of the system.

Only the first of these steps was taken by Dirac, who relied instead on the equivalence of the resulting Hamiltonian with that obtained in (1.2.2). That we shall consider in (1.2.3). We conclude with the expression of Eq.(15) in terms of the number operators.

Dirac introduced the number and phase operators in the manner indicated in the introduction, quoted in (1.2.1), and through his method 3, the second quantization process. The essential properties, that in fact provide the best definition of these operators, are:

$$\begin{aligned}(N_r+1)^{1/2}\, e^{-i\theta_r/\hbar} &= e^{-i\theta_r/\hbar}\, N_r^{1/2}\\ N_r^{1/2} e^{i\theta_r/\hbar} &= e^{i\theta_r/\hbar}\, (N_r+1)^{1/2}\end{aligned} \tag{16}$$

which follows from the (ill-defined) CCR's:

$$\theta_r N_s - N_s \theta_r = i\hbar. \tag{17}$$

for which Dirac gave the explicit representation:

$$\begin{aligned}N_s f(n_1,n_2,\ldots,n_r,\ldots) &= n_s f(n_1,n_2,\ldots,n_r,\ldots)\\ \theta_r f(n_1,n_2,\ldots,n_r,\ldots) &= i\hbar\frac{\partial}{\partial n_r} f(n_1,n_2,\ldots,n_r,\ldots)\end{aligned} \tag{18}$$

for these operators, acting on an arbitrary function of the n's. These formulae led him into a systematic error that we have already discussed; for it is elementary to establish that:

$$\begin{aligned}e^{\pm i\theta_r/\hbar} f(n_1,n_2,\ldots,n_r,\ldots) &= e^{\mp\partial/\partial n_r}\, f(n_1,n_2,\ldots,n_r,\ldots)\\ &= f(n_1,n_2,\ldots n_r \mp 1,\ldots)\end{aligned} \tag{19}$$

from which he concluded that the *creation* operator is given by $e^{-i\theta_r/\hbar}$ and the *annihilation* operator[7] by $e^{i\theta_r/\hbar}$. But a glance at Eq.(16) shows that it is the other way round. The error lies in the interpretation of Eq.(19); Dirac appeared to conclude that, for example, $f(n_1,n_2,\ldots,n_r+1,\ldots)$ describes a system with an additional particle in the state

[7] Strictly speaking neither of these operators are creation and annihilation operators, which are rather given by $\sqrt{N}e^{i\theta/\hbar}$ and $e^{-i\theta/\hbar}\sqrt{N}$ resepectively.

r. But this is not so. As formulated, f is not an eigenfunction of the number operators with eigenvalues $n_1, n_2, \ldots$ etc., but simply a function of these variables (see also footnotes 3 and 6).

Eq.(19) is true[8] for any function of the n's and in particular it holds true when f is replaced by the expansion coefficients $b_{n_1 n_2 \ldots}$, regarded as functions of the n's. We may therefore write:

$$b_{n_1 n_2 \ldots n_r - 1, \ldots, n_s + 1 \ldots} = e^{i(\theta_r - \theta_s)/\hbar} b_{n_1 n_2 \ldots n_r \ldots n_s \ldots}$$

and replace *e.g.* $n_r b_{n_1 n_2 \ldots n_r \ldots}$ by $N_r b_{n_1 n_2 \ldots n_r \ldots}$ and in this way finally obtain:

$$i\hbar \dot{b}_{n_1 n_2 \ldots}(t) = \sum_r \sum_s \langle u_r, H u_s \rangle N_r^{1/2} (N_s + 1 - \delta_{rs})^{1/2} e^{i(\theta_r - \theta_s)/\hbar} b_{n_1 n_2 \ldots}(t) \tag{20}$$

The Hamiltonian is

$$H = \sum_r \sum_s \langle u_r, H u_s \rangle N_r^{1/2} (N_s + 1 - \delta_{rs})^{1/2} e^{i(\theta_r - \theta_s)/\hbar} \tag{21}$$

and makes no reference to the number of particles present. It has the elegant interpretation, that applied to an n-particle state, it annihilates a particle in the state s and creates a particle in the state r, with amplitude proportional to the 1-particle matrix element between these two states, in such a way as to make no distinction between *which* particle is in that state.

This use of number and phase operators is objectionable from a mathematical point of view as we have already indicated.

Instead of appealing to the CCR's, Eq.(17), or the concrete representation of Eq.(18), one can simply *define* the operators N_r and $e^{\pm i\theta_r/\hbar}$ by Eq.(16), and represent the action of both operators concretely on Fock space. In the present context it is perhaps more germane to define these

[8] Dirac's interpretation is incorrect, his formulae are based on mathematically inconsistent assumptions, but he gets the right equations out. (This is quite typical, and physics is a mystery); we shall shortly establish the following equations on a rigorous basis.

operators concretely, and then check that they satisfy Eq.(16)[8]; to that end we define $\exp(-i\theta_r/\hbar)$ as the operator:

$\exp(-i\theta_r/\hbar)$: $\mathcal{H}^n \longrightarrow \mathcal{H}^{n-1}$

given by the action on the basis vectors $u_{r_1} \otimes u_{r_2} \otimes \ldots \otimes u_{r_n}$ (which are clearly dense in $\mathcal{H}^n$):

$$\exp(-i\theta_r/\hbar): u_{r_1} \otimes u_{r_2} \otimes \ldots \otimes u_{r_n} \longrightarrow \langle u_r, u_{r_1} \rangle u_{r_2} \otimes u_{r_3} \otimes \ldots \otimes u_{r_n} \qquad (22)$$

It is elementary to prove that $\exp(-i\theta_r/\hbar)$ maps a completely symmetric state in $\mathcal{H}^n$ into a completely symmetric state in $\mathcal{H}^{n-1}$, and further that its adjoint restricted to the symmetric subspace of $\mathcal{H}^n$ is a map onto the symmetric subspace of $\mathcal{H}^{n+1}$. We denote $\exp(-i\theta_r)^*$ by $\exp(i\theta_r)$ and note that their action in the occupation number representation is:

$$\exp(\pm i\theta_r/\hbar): u_{n_1 n_2 \ldots n_r \ldots} = u_{n_1 n_2 \ldots n_r \pm 1 \ldots}$$

and that indeed Eq.(16) is satisfied. The (conventional) creation and annihilation operators are now *defined* as:

$$a(u_r) = (N_r+1)^{1/2}\exp(-i\theta_r/\hbar), \qquad a^*(u_r) = N_r^{1/2}\exp(i\theta_r/\hbar) \qquad (23)$$

(on the LHS we have introduced standard notation; we shall not use it for the time being). If everywhere Dirac's use of the operator $e^{\pm i\theta_r/\hbar}$ is replaced by $\exp(\pm i\theta_r/\hbar)$ a mathematically well-defined formula results, precisely equivalent in meaning to that intended by Dirac, with the exception of his method 1 (i.e. the theory of (1.2.3)). This strategy is therefore of questionable value in the context of his equivalence proof, that we now consider.

[9] The general question, of whether one proceeds from concrete representations or (abstractly) from an algebraic viewpoint is a principle concern of this thesis. At this point it is appropriate to consider the concrete definition, because the theory of this subsection is (effectively) concretely defined (even though we have not specified the Hilbert space, or the Hamiltonian). This will become clear in Part 2. The claims made in the immediate sequel, and the general strategy used, is due to Fock [1932], and Cook [1953]. Their contributions involve no fundamental conceptual modifications in the theory, which were not introduced in the physics literature, and we do not consider these papers in this thesis. An expositary summary is given in (1.3.4).

So far we have considered the interaction Hamiltonian V (which enters into H in Eq.(20)) as an external interaction to which each particle in the assembly is subject. Dirac now considered explicitly introducing a new physical system with free Hamiltonian W which couples with the assembly of bosons through the interaction Hamiltonian V. He considered the new physical system (the *perturbing system*) described by the action-angle variables J,ω , where W is a function of J only, and where V is a function of both J and ω and all of the dynamical variables which describe the boson assembly.

Clearly the total Hamiltonian H_T is just:
$H_T = H + W = F + V + W.$
Concerning V he made the natural assumption that it has vanishing matrix elements between states where more than one boson makes a transition. This perturbation cannot any longer be defined as the second quantization of a 1-particle perturbation, for obvious reasons, but this assumption implies that the Hamiltonian nevertheless has the same form as Eq.(15).

What follows requires a rather extensive reconstruction, if we are to maintain contact with Hilbert space theory. I shall first describe what Dirac did, and then reformulate his treatment in Hilbert space.

Dirac wrote down Eq.(9), modified only by the inclusion of a summation over the eigenvalues of the action variable for the perturbing system, that is:

$$i\hbar\dot{b}(j,r_1r_2\ldots) = \sum_{j',s_1,s_2,\ldots} H_T(jr_1r_2\ldots;j's_1s_2\ldots)b(j's_1s_2\ldots) \qquad (24)$$

and declared:

> The matrix element $H_T(j,r_1r_2\ldots;j',s_1s_2\ldots)$ is now always a constant. As before, it vanishes when more than one s_n differs from the corresponding r_n. When s_m differs from r_m and every other s_n equals r_n it reduces to $H(jr_m;j's_m)$, which is the $(jr_m;j's_m)$ matrix element (with the time

factor removed) of $H = F + V$, the proper energy plus the perturbation energy of a single system of the assembly; while when every s_n equals r_n, it has the value $H_P(j)\delta_{jj'}$ + $\Sigma_n H(jr_n; j'r_n)$. If, as before, we restrict the eigenfunctions symmetrical in the variables $r_1, \ldots$ we can again transform to the variables $n_1, n_2, \ldots,$ which will lead, as before, to the result

$$i\hbar \dot{b}_{j,n_1n_2\ldots} = W(j)b_{j,n_1n_2\ldots} \qquad (25)$$

$$+\Sigma_{j'}\Sigma_r\Sigma_s n_r^{1/2}(n_s+1-\delta_{rs})^{1/2}H(jr;j's)b_{j',n_1n_2\ldots n_r-1\ldots n_s+1\ldots}$$

This is the Schrödinger equation corresponding to the Hamiltonian function

$$H_T = W(J) + \sum_{r,s} H_{rs} N_r^{1/2}(N_s+1-\delta_{rs})^{1/2}e^{i(\Theta_r-\Theta_s)/\hbar} \qquad (26)$$

in which H_{rs} is now a function of the J's and ω's, being such that when represented by a matrix in the (J) scheme its (jj') element is H(jr;j's). (It should be noticed that H_{rs} still commutes with the N's and Θ's.) [255-6].

Let me expand on Dirac's treatment. It is clearly necessary to construct the Hilbert space: $\mathcal{H}_T = \mathfrak{h}_P \otimes \mathcal{H}$ where $\mathfrak{h}_P$ is the Hilbert space of the free perturbing system with inner product $\langle .,. \rangle_P$, spanned by the basis states w_j which are eigenstates of the Hamiltonian W (therefore eigenstates of the action operator). We denote these states w_j, $w_{j'}$, etc. and use the notation $u_{j,n_1n_2\ldots}$ for the state $w_j \otimes u_{r_1} \otimes u_{r_2} \otimes \ldots$, and similarly for the occupation number states (and expansion coefficients). The natural inner product on $\mathcal{H}_T$ is then:

$$\left(u_{j,n_1n_2\ldots}, u_{j',n_1'n_2'\ldots} \right)_T = \langle u_j, u_{j'} \rangle_P \left(u_{n_1n_2\ldots}, u_{n_1'n_2'\ldots} \right) .$$

We formulate Dirac's assumptions concerning the dependence of W and V on the variables of the perturbing system and the system of bosons as follows; we write

$$W = H_P \otimes \mathbb{I} \otimes \mathbb{I} \otimes \ldots \otimes \mathbb{I} \qquad (27)$$

(n+1 factors in all), where H_P acts on $\mathfrak{h}_P$ only, so that:

$$(u_{j,n_1n_2\ldots}, Wu_{j',n_1'n_2'\ldots})_T = \langle u_j, Wu_{j'} \rangle_P \delta_{n_1n_1'}\delta_{n_2n_2'}\ldots$$

and following the same arguments which led from Eq.(9) to Eq.(12) we may conclude:

$$i\hbar\, b_{j,r_1r_2\ldots}(t) = \langle w_j, Ww_j\rangle_P\, b_{j,r_1r_2\ldots}(t)$$

$$+\sum_{j'}\sum_k \sum_{s_k\neq r_k} \langle u_{j,r_k}, H(k) u_{j',s_k}\rangle_I\, b_{j',r_1r_2\ldots r_{k-1}s_kr_{k+1}\ldots}(t)$$

$$+\sum_{j'}\sum_k \langle u_{j,r_k}, H(k) u_{j',r_k}\rangle_I\, b_{j',r_1r_2\ldots}(t).$$

where $\langle .,.\rangle_P$ is the inner product on $\mathfrak{h}_P$ and $\langle .,.\rangle_I$ is the inner product on $\mathfrak{h}_P\otimes\mathfrak{h}$. By procedures identical to the foregoing we may write this in the occupation number representation as:

$$i\hbar\,\dot{b}_{j,n_1n_2\ldots} = \langle w_j, Ww_j\rangle_P b_{j,n_1n_2\ldots} +$$

$$\sum_{j'}\sum_r\sum_s n_r^{1/2}(n_s+1-\delta_{rs})^{1/2}\langle u_{j,r}, Hu_{j',s}\rangle_I b_{j',n_1n_2\ldots n_r-1\ldots n_s+1\ldots}$$

which is clearly what Dirac had in mind in Eq.(25). However passing from this equation to Eq.(26) is not straightforward, because Eq.(25) is expanded in terms of the basis states $u_{j,n_1n_2\ldots}$ of $\mathcal{H}_T$, not the $u_{n_1n_2\ldots}$ which span $\mathcal{H}$, yet we have factored off the j dependence into the "2-particle" (perturbing system plus 1-boson) inner product $\langle .,.\rangle_I$. That is, $\langle u_{j,r}, Hu_{j',s}\rangle$ is a c-number, but it must be summed over the basis states of $\mathfrak{h}_P$. It is, in other words, an operator on $\mathfrak{h}_P$ which is basis dependent, yet this is not explicit in Eq.(26). To remove this anomaly we take the matrix elements of H only with respect to the basis u_r, that is we define the operator on $\mathcal{H}_T$

$$H_{rs} = \langle u_r, Hu_s\rangle \otimes \mathbb{I}\otimes\ldots.\otimes\mathbb{I}$$

(where $\langle u_r, Hu_s\rangle$ is an operator on $\mathfrak{h}_P$, in Dirac's words "a function of the J's and ω's"). With this and the operator W in the form of Eq.(27) we may now apply the transformation theory in an unrestricted way with respect to $\mathcal{H}_T$.

We have that:

$$b_{j'n_1n_2\ldots n_r-1\ldots n_s+1\ldots}(t) = (u_{j',n_1n_2\ldots n_r-1\ldots n_s+1\ldots}, \psi(t))_T$$

$$= (e^{i(-\Theta_r+\Theta_s)/\hbar}u_{j,n_1n_2\ldots}, \psi(t))_T$$

$$= (u_{j,n_1n_2\ldots}, e^{i(\Theta_r-\Theta_s)/\hbar}\psi(t))_T$$

(had Dirac written down this equation the error of his interpretation of the $e^{\pm ie'/\hbar}$'s would have been evident), so that we may write Eq.(24) as:

$$i\hbar\frac{\partial}{\partial t}\left(u_{j,n_1n_2\ldots},\psi(t)\right)_T =$$

$$\sum_{j',n_1'n_2'\ldots}\left(u_{j,n_1n_2\ldots},\left[\;W+\sum_{r,s}H_{rs}N_r^{1/2}(N_s+1-\delta_{rs})^{1/2}\right]\;u_{j',n_1'n_2'\ldots}\right)_T$$

$$\cdot\left(u_{j',n_1'n_2'\ldots},\,e^{i(\Theta_r-\Theta_s)/\hbar}\,\psi(t)\right)_T$$

By the assumed completeness of the $u_{j,n_1n_2\ldots}$'s Eq.(20) follows.

We can separate off the contribution from the free (1-boson) Hamiltonian F in $H_{rs} = F_{rs} + V_{rs}$ since this has vanishing matrix elements between states of different quantum numbers; that is, we can write:

$$H_{rs} = \langle u_r, Hu_s\rangle\otimes\mathbb{I}\otimes\ldots = \langle u_r, Fu_s\rangle\otimes\mathbb{I}\ldots + \langle u_r, Vu_s\rangle\otimes\mathbb{I}\otimes\ldots$$
$$= E_r\delta_{rs}\,\mathbb{I}\otimes\mathbb{I}\otimes\ldots + V_{rs}$$

or

$$H_{rs} = E_r\delta_{rs} + V_{rs}$$

so that finally:

$$H_T = W(J) + \sum_r E_rN_r + \sum_{r,s}V_{rs}N_r^{1/2}(N_s+1-\delta_{rs})^{1/2}e^{i(\Theta_r-\Theta_s)/\hbar} \qquad (28)$$

This Hamiltonian describes the quantum mechanics of an n+1 body problem, in which the n-particles obey Bose-Einstein statistics. There is therefore no question of an equivalence with the QED of (1.2.2), which describes single photon emission and absorption processes. That H_T can only describe transitions which conserve the total particle number, is formally a consequence of the fact that the Hamiltonian contains only bilinear combinations of the creation and annihilation operators. From the point of view of the quantum mechanics (i.e. the equivalent Hamiltonian in the usual formulation of Eq.(24)) this is not a formal question, but a structural feature of the theory. It is in this sense that with the foregoing theory, we approach a QFT, because the particle number is no longer structurally entrenched in the dynamical framework, and we can easily envisage a Hamiltonian in which the creation and annihilation operators are *not* always paired together in

this way[10].

Nevertheless in application to photons, Dirac was able to effect a correspondence with the QED. He hit on the simple idea that a photon which does not exist can still be described as existing, but with zero frequency and energy. Obviously this is not possible for any particle of non-zero rest mass. (It is an unfortunate complication, that a non-relativistic theory must then be applied to such an intrinsically relativistic particle as the photon; on the other hand it could not be otherwise, since a classical field theory must - from an historical point of view - be long-range, hence massless).

Dirac formulated the idea as follows:

> Since there is no limit to the number of light-quanta that may be created in this way, we must suppose that there are an infinite number of light-quanta in the zero state, so that the N_o of the Hamiltonian is infinite. We must now have θ_o, the variable canonically conjugate to N_o, a constant, since $d\theta_o/dt = \partial H_T/\partial t = E_o$ + terms involving $N_o^{-1/2}$ or $(N_o+1)^{-1/2}$, and E_o is zero. In order that the Hamiltonian to remain finite it is necessary for the coefficents V_{ro} and V_{or} to be infinitely small. We shall suppose that they are infinitely small in such a way as to make $V_{ro}N_o^{1/2}$ and $V_{or}N_o^{1/2}$ finite, in order that the transition probabilities may be finite. Thus we put

[10] This is to characterize a QFT in terms of the description of particle creation and annihilation processes; I do not wish to characterize a QFT theory in this way, but in effecting a correspondence with the QED it is the major obstacle. Incidentally, particle number is still structurally entrenched in the theory - only through the fact that the Hilbert space is fixed as the n+1 fold tensor product on 1-particle Hilbert spaces. A Hamiltonian of the more general kind considered, would not be an operator on this space. Obviously taking the direct sum of all such Hilbert spaces (that is for all n) eliminates this difficulty, which yields Fock space. The point here is that the operator expression of dynamical law developed by Dirac is indifferent to particle number, which enters as a variable, and the Hamiltonian makes perfect sense as an operator on Fock space. That is not the case with the mechanical formalism, obviously.

$$V_{ro}(N_o+1)^{1/2}e^{-i\theta_o/\hbar} = V_r$$
$$V_{or}N_o^{1/2}e^{i\theta_o/\hbar} = V_r^* \qquad (29)$$

The foregoing passage is, to borrow a phrase of Segal's, a kind of mathematical poetry. Dirac here presents a challenge to the mathematics community that has not, in more than half a century, come anywhere near to finding a viable solution. Nevertheless, through this magnificent argument, Dirac was able to obtain his goal: we have only[11] to set $V_{rs} = 0$, for $r>0$, $s>0$, (or both zero), substitute Eq.(29) in Eq.(28) to obtain:

$$H_T = W(J) + \sum_r E_r N_r + \sum_r V_{ro} N_r^{1/2} e^{i\theta_r/\hbar} + V_{or}^* (N_r + 1)^{1/2} e^{-i\theta_r/\hbar}$$

and then set

$$V_r = V_s = \hbar^{1/2} c^{-3/2} (\nu_r/\sigma_r)^{1/2} \dot{x}_r$$
$$E_r = h\nu_r$$
$$W(J) = H_o$$

to obtain precisely the QED Hamiltonian, Eq.(7). With this, Dirac declared:

> The wave point of view is thus consistent with the light-quantum point of view and gives values for the unknown interaction coefficient V_{rs} in the light-quantum theory. These values are not such as would enable one to express the interaction energy as an algebraic function of canonical variables. [263].

Von Neumann, in his account of this equivalence proof[12], was

[11] Dirac remarked: "Since the wave theory gives $V_{rs} = 0$ for r, $s \neq 0$, it would seem to show that there are no direct scattering processes, but this may be due to an incompleteness in the present wave theory."[263]. Shortly after (Dirac [1927b]) he indeed established that this is due to an incompleteness - the neglect of higher order terms in the expansion of the relativistic Hamiltonian equation, of the form $(p_i - eA_i)^2 - m^2c^2 = 0$, of which only the leading term yields the non-relativistic Hamiltonian.

[12] I shall not discuss von Neumann's account in any detail. Most of the clarifications which he introduced are contained in the foregoing, although I was not aware of his study at the time of writing. The principal exception is his treatment of the creation and annihilation operators, which followed that of Jordan and Klein [1927], and has since become standard in the context of field quantization (essentially Dirac's method 3).

less restrained:

> In this way one of the most difficult paradoxes of the earlier form of the quantum theory, the dual nature of light (electromagnetic waves and discrete corpuscules or light quanta), is brilliantly resolved. To be sure, it is difficult to find a direct, clear-cut interpretation of the interaction energy V ....nevertheless, we can accept this with the interpretation that each model-description is only an approximation, while the exact content of the theory is furnished solely by the expression for the H operator (i.e. H_T of Eq.(7)). (Von Neumann [1932 p.282-3]).

In what sense does the Dirac equivalence resolve the problem of wave - particle duality? Von Neumann means something quite specific, as is clear from his subsequent comment:

> It has often been said that the quantum mechanics involves the same dual nature, since the discrete particles (electrons, protons) are also described by wave functions, and exhibit typical wave properties, i.e., diffraction by a grating... . In contrast with this, however, it is to be noted that quantum mechanics derives both "natures" from a single unified theory of the elementary phenomena. The paradox of the earlier quantum theory lay in the circumstance that one had to draw alternately on two contradictory theories (electromagnetic theory of Maxwell-Hertz, light quantum theory of Einstein) for the explanation of the experience.

Von Neumann appears to suggest that the NRQM is also a "single unified theory" of the elementary phenomena (protons and electrons, and their dual nature). Would then a relativistic quantum mechanics of the photon be just as good as the Dirac equivalence? If not why not? I suggest there are two lines of argument: first, the Dirac equivalence suggests that there are *two* classical limits to the theory, classical electromagnetic field theory, and a classical many-particle theory. The NRQM of the electron or proton presumably only has the one (particle) classical limit. Second, and closely related, the Dirac equivalence permits a *mathematical* interpretation at the *quantum* level of the theory, from either the particle or the field view-point. That is to say, whilst we may attempt to interpret the NRQM from a field view-point, the interpretation is *ad hoc* and free-hanging for from a mathematical point of view the theory is built on the particle concept alone. In particular

on this second line the quantum particle is an excitation of the field, which generates an altogether new heuristic in the interpretation of the theory. In an important respect, it is simple to understand how a field theory can acquire particulate characteristics (through the quantization of energy), and more difficult to understand how a particle theory can acquire field characteristics (through the superposition principle).

This second strategy will be a theme throughout this thesis[13]; but on the surface it has nothing to do with the wave-particle duality associated with the (1-particle) wave mechanics. In particular the electromagnetic field equations from which Dirac developed the many-particle interpretation is inhomogeneous and has no direct connection with a Schrödinger equation for the photon. It is clear that for just this reason the equivalence proof is essentially figurative; in the interacting case, with a single and remarkable exception (Coulomb non-relativistic coupling) there is no rigorous equivalence of field and many-particle system. However from the point of view of the wave-particle duality it is already significant if a field many-particle equivalence exists, which can be directly related to the wave-particle duality in the 1-particle theory. That is the case for the free non-relativistic theory, as we shall see; local densities in the quantum fields are formally identical to the local probability densities of the 1-particle theory. In this situation we may interpret these densities as the expectation values of densities in the associated quantum field theory; I shall formulate this interpretation as a modification of the Schrödinger "electromagnetic" interpretation, where in place of Schrödinger's classical c-number fields we have quantum fields instead.

All of this follows from Dirac's method 3.

[13] I shall not consider the classical limit (in the usual sense $\hbar \longrightarrow 0$) of a QFT at all; in Section 3.5 a different approach to classical limits in developed.

Section 1.3 Non-Relativistic Quantum Field Theory and the Field - Many-particle Equivalence

> While doing this work, I got one of those ideas out of the blue, namely to take the Schrödinger wave equation and apply a process of quantization to the wave function itself. The wave function was previously always considered as expressed by ordinary numbers, c-numbers. What would happen if you turned them into q-numbers, and assumed that they are noncommuting with their conjugates? That led to a theory which was equivalent to the theory of radiation which I had been setting up, and provided an alternative way of introducing the subject. It gave rise to a procedure which has become known as second quantization.
>
> P. Dirac

1.3.1. Method 3: second quantization (field quantization)

Dirac did not present his method 3 as a quantization of a classical field theory, and had he done so, the terminology "second quantization" would never have arisen. This is a shame, because the terminology is very misleading. As Rosenfeld was to remark with some exasperation, many years later:

> ...as ought to be well known,.what is misleadingly called "second quantization" is nothing else than an equivalent formulation of (the) scheme (of quantum mechanics) with the help of convenient operators in (the occupation number representation). (Rosenfeld [1963 p.355].)

In other words, Rosenfeld claims that "second quantization", in customary parlance (*circa* 1960; things have not changed so much since then), means Dirac's method 2. I think Rosenfeld is probably right on this score, but certainly in the early period that we here consider second quantization ment either method 2 or method 3 (the two were rarely

distinguished)[1]. In this situation I shall systematically use the phrase "canonical second quantization" for the method 2; unless so qualified, the meaning of the term "second quantization" will depend on the context. In the present context, it means a *field quantization*, although Dirac himself carefully abstained from placing any clear interpretation on the formalism which he invented.

In the introductory section, he did however carefully distinguish "between a light-wave and the de Broglie or Schrödinger wave associated with the light-quanta" [p.247]. In this way we learn that he did *not* distinguish the Schrödinger wave from the de Broglie wave, which is essentially the point at issue, in the context of distinguishing the Schrödinger wave mechanics from a classical field theory[2]. (I shall systematically use the terminology "de Broglie field" in this connection, that is, to denote the classical matter field with field equation formally identical, in the 1-particle case, to the Schrödinger equation in 3+1 dimensional space-time).

When he came to actually develop the method 3, he referred to the method of his semi-classical radiation theory (the interaction picture); in this paper he sometimes referred to

[1] In the early 30's there was no such thing as "a" quantum field theory; there was the Dirac theory of the electron, the Dirac electron theory in second quantized form, the Heisenberg-Pauli theory of radiation, and so on. The idea that there may exist classical matter fields, that had not already been studied in the classical era, did not really intrude until the Pauli-Weiskopf theory and the Yukawa theory of the meson. In this context Dirac's method 3 was considered, I suggest, essentially a method for carrying through his method 2.

[2] Is this anachronistic? In modern times de Broglie waves usually refer to the idea of classical matter fields, see, e.g. Tomonaga's book [1962] on the (non-relativistic) field - many-particle equivalence, formulated in terms of the equivalence of a quantized de Broglie field with the many-particle wave mechanics. The difficulty here is that Schrodinger as well as de Broglie championed a matter wave interpretation. On balance I consider that the distinction was, however, reasonably current at the time of Dirac's writings.

(what ought to have been) the 1-particle wave function as describing a particle ensemble. Indeed, he went on to say that the previous theory

> ...gave immediately the probable number of systems in that state at that time for an assembly of the systems that are independent of one another and are all perturbed in the same way. The object of the present section is to show that the equations for the rates of change of these probable numbers can be put in the Hamiltonian form in a simple manner, whch will enable further developments in the theory to be made. [p.248].

I shall briefly describe Dirac's procedure, indicating in square brackets the appropriate terminology, viewing the method as a field quantization. First, he gave two c-number action-angle formulations of the Schrödinger equation [de Broglie field equation], the latter of which he then took to define a q-number Hamiltonian [field Hamiltonian] (by simply considering the action-angle variables as q-numbers, the number and phase q-numbers). He then considered the Hamiltonian operating on a wave-function, [state of the de Broglie field], a function of all the number eigenvalues, and in the case in which all these are zero but one (which is unity) verified the equivalence with the usual 1-particle wave equation. He then went on to develop his method 2, and from the formal identity of the Schrödinger equation in this case (for an n-boson ensemble) with the previously obtained equation [Schrödinger equation for the quantized de Broglie field, with the total field energy as Hamiltonian], concluded that "the [field] Hamiltonian describes the effect of a perturbation on an assembly satisfying the Einstein-Bose statistics." [p.255].

We shall not bother with the first (classical) formulation of the theory in the interaction picture, which Dirac did not quantize, but go direct to the second formulation, which is that of (1.2.3) Eq.(8) *et seq*[3] . We have the Hamiltonian

[3] For the sake of definiteness, I use the same notation as in (1.2.3) and as there cast the theory into Hilbert space form. Interpreting this theory as a classical field theory, the Hilbert space $\mathfrak{h}$ may be considered the classical solution manifold.

$H = F + V$

and the Schrödinger equation

$H\,\psi(t) = i\hbar\partial\psi/\partial t$

which takes the form:

$$i\hbar\,\dot{b}_k = \sum_j H_{kj} b_j \qquad (1)$$

where

$H_{jk} = \langle u_j, Hu_k \rangle, \quad Fu_k = E_k u_k$

$\psi(t,\mathbf{x}) = \sum_k b_k(t) u_k(\mathbf{x})$

as before. The u_k's are time-independent, and Eq.(1) is the Schrödinger equation for the expansion coefficients of the time-dependent wave function with respect to the orthonormal basis provided by the u_k's; viewed as a classical field theory, this procedure is standard, but we say rather that the expansion is a Fourier decomposition of the field (with "box" normalization or the use of improper energy eigenstates, the two are formally equivalent).

According to Dirac[4], the b_k's are normalized to unity, but [p.248]:

> the theory will apply directly to an assembly of N similar independent systems if we multiply each of these b_k's by $n^{1/2}$ so as to make $\Sigma_k |b_k|^2 = n$. We shall now have that $|b_k|^2$ is the probable number of systems in the state k.

Note that considered as a classical matter theory, this probability is interpreted as the total *mass* of the associated field configuration.

Dirac now noted that if we consider b_k, $i\hbar\overline{b_k}$ as canonically conjugate variables, where $\overline{b_k}$ obeys the complex conjugate equation to Eq.(1), *i.e.*

$$-i\hbar\,\dot{\overline{b}}_k = \sum_j \overline{H_{kj} b_j} = \sum_j \overline{b}_j H^*_{jk} = \sum_j \overline{b}_j H_{jk}$$

(H is assumed Hermitian), then both this and Eq.(1) can be put into Hamiltonian form with the Hamiltonian:

$$\hat{H} = \sum_{jk} \overline{b}_j H_{jk} b_k$$

[4]Dirac made this comment in connection with the interaction picture expansion coefficients, but the same applies in the Schrödinger picture.

That is:

$$db_k/dt = \frac{1}{i\hbar}\,\partial\hat{H}/\partial t\ , \quad i\hbar\, d\bar{b}_k/dt = -\,\partial\hat{H}/\partial t$$

For orientation, the b's, $\bar{b}$'s become the usual Heisenberg-picture momentum-space annihilation and creation operators upon quantization. In the non-relativistic theory, these are identified (up to a factor in $\hbar$) with the canonically conjugate momentum space (Heisenberg picture) fields. The theory appears incomplete only because he has failed to write down the Lagrangian and from this obtain the Hamiltonian, canonically conjugate variables, and so on.

However Dirac now introduces the canonical action-angle variables by the contact transformation:

$$b_i = n_i^{1/2} e^{-i\Theta_i/\hbar}, \quad \bar{b}_i = n_i^{1/2} e^{i\Theta_i/\hbar}$$

in terms of which the Hamiltonian $\hat{H}$ is written:

$$\hat{H} = \sum_{ij} H_{ij}\, n_i^{1/2} n_j^{1/2} e^{i(\Theta_i-\Theta_j)/\hbar}$$

He now quantizes this system. In fact he actually writes down the CCR's for the b's and b^*'s:

$$b_i . i\hbar b_j^* - i\hbar b_j^* . b_i = i\hbar\,\delta_{ij}$$
$$[b_i, b_j] = [b_i^*, b_j^*] = 0 \qquad (2)$$

(which would effect the customary Lagrangian quantization) but proceeds directly to the relationships Eq.(16) of (1.2.3) noting that

$$b_i = (N_i+1)^{1/2} e^{-i\Theta_i/\hbar} = e^{-i\Theta_i/\hbar}$$

$$b_i^* = N_i^{1/2}\, e^{i\Theta_i/\hbar} = e^{i\Theta_i/\hbar}\,(N_i+1)^{1/2}.$$

The Hamiltonian $\hat{H}$ becomes:

$$\hat{H} = \sum_{ij} b_i^*\, H_{rs}\, b_s = \sum_{ij} N_i^{1/2} e^{i\Theta_i/\hbar} H_{ij} (N_j+1) e^{-i\Theta_j/\hbar}$$

$$= \sum_{ij} H_{rs} N_r^{1/2} (N_s^{1/2}+1 - \delta_{rs})\, e^{i(\Theta_r-\Theta_s)/\hbar}$$

Noting that the H_{rs} are still c-numbers, he applies this Hamiltonian to a wave function $\psi(n_1, n_2, \ldots)$ using Eq.(19) of (1.2.3), to obtain Eq.(15); deriving the latter as described in (1.2.3), establishes the equivalence of the present method (method 3) with method 2. Since Dirac did not consider the present theory a field quantization (but rather, as seems most likely on the evidence, a mathematical mnemonic), h considered the equivalence as establishing

the correct interpretation for the foregoing theory. It was left to Pascual Jordan to draw the obvious inferences.

1.3.2. Lagrangian field quantization.

So obvious, indeed, that Jordan may be forgiven the comment:

> Dirac has recently shown how Einstein's conjecture, that the ideal material gas stands in analogy to the light quantum gas through the quantization of waves in ordinary 3-dimensional space, follows from the exact quantum mechanics... (Jordan [1927b p.473]).

Jordan had long been committed to the idea that one should apply the new mechanics to the radiation field itself; unlike the rest of the Göttingen group, he had contributed to the development of the Einstein gas theory[5] and had systematically applied the old quantum theory to the radiation field itself[6]. His work on the thermodynamic equilibrium of bosons in 1925 led to an interest in the Eddington theory of particle (pair) annihilation, as modified by Otto Stern in 1926 to include pair creation processes[7]. In particular he considered the conditions for thermodynamical equilibrium for a grand canonical ensemble of massive bosons (particle number variable) subject to pair creation and annihilation (Jordan [1927a]). His first important contribution to QFT appeared in 1927, "Zur Quantenmechanik der Gasentartung", written immediately following this paper; in its mere 8 pages can be found almost all of the ideas relevant to fermion field quantization. For Jordan (*cf.* the discussion of (1.1.4)) the

[5] He generalized the Einstein gas theory to the case of zero mass, and where the total particle number is not conserved (Jordan [1925]).

[6] This was the subject of his doctoral dissertation. Subsequently published in [1924], this work was almost immediately to become obsolete. In his collaboration with Born, and Born and Heisenberg, he was responsible for the sections devoted to QFT ((1.1.2)).

[7] treated semi-classically and on the fundamental heuristic of the equivalence of mass and energy. Both were concerned with models of stellar evolution (Eddington [1926], Stern [1925], [1926]).

significance of the equivalence of Dirac's method 2 and 3 must have lept out of the page:

> I have been extremely thankful to Dirac...my idea that the solution of the vexing problem of Einstein's light quanta might be given by applying quantum mechanics to the Maxwell field itself, aroused the doubt, scepticism and criticism of several good friends. But one day I visited Born, he was reading a new publication of Dirac, and he said: "Look here, what Mr. Dirac does now. He assumes the eigenfunctions of a particle to be non-commutative observables again." I said: "Naturally." And Born said: "How can you say "naturally"?" I said: "yes, that is, as I have asserted repeatedly, the method which leads from the 1-particle problem to the many-body problem in the case of Bose statistics." (Mehra [1973 p.296]).

I shall not discuss the "Gasentartung" paper in any detail; it is enough to say, that it contained a reworking of the analysis of the Einstein fluctuation formula using Dirac's number-phase operators, and a (flawed) proof of the methods 3 = 2 equivalence for fermions, conducted via an elaborate application of his transformation theory to the number and phase operators. In the third and final section, entitled "final remarks", he declared:

> The results obtained here leave it scarcely to be doubted that - in spite of the validity of the Pauli over the Bose statistics for electrons - a quantum mechanical wave theory of matter can be formulated, in which electrons can be described as quantized waves in ordinary 3-dimensional space. Furthermore, the empirically appropriate (naturgemaße) formulation of quantumelectrodynamics will be achieved through the conception of light and matter equally as interacting waves in 3-dimensional space. The fundamental basis of the electron theory, the existence of discrete charged particles, is thereby established as a characteristic quantum effect, coextensive (gleichbedeutend) with the appearance of matter waves only in discrete quantum states. The Schrödinger eigenfunction for material particles constructed by Dirac and Heisenberg plays a rôle in the space of this picture which is in no sense an analogy to electromgnetic waves. On the contrary, they prove to be a special case of the general probability amplitude, which is employed as an auxilliary mathematical device to the description of the statistical behaviour of quantized light and matter waves. (Jordan [1927b p.480]).

This short and closely written passage constitutes a remarkable document: it is at once a historic challenge to the physics community, the declaration and broad definition

of a research programme that was to dominate foundational research into the structure of matter for three generations. And it was also a metaphysical statement, an assertion of the things that exist in the world: not the matter waves of Schrödinger, still fighting his lone and doomed battle for the classical world view, nor the particles of Heisenberg, their properties veiled by the disturbance theory of measurement, but a hybrid of the two, particles as the count of occupied states determined by the dynamics of an interacting quantum field theory.

That Jordan essentially got it right was all the more remarkable in view of the fact that no-one, at the time, knew how to incorporate "interacting waves" into the second quantization process (field quantization). The very term was ambiguous: Hertz's "Electric Waves" had run into its 3rd edition by 1914 and continued to be used up to the 1940's in the Gymnaseums; a German physicist would have been familiar with the continuum treatment of electric current. Yet Lorentz's writings were by far the most familiar by the 1920's with their attendant dualist model - still the locus of contemporary research by such authorities as Schwarzchild, Abraham, Mie, Weyl, and Poincaré - and amongst them there was no consensus as to the nature of the interaction of field and electron. To what classical model should one appeal? Certainly the Dirac QED (method 1), a quantized dualistic theory in the Lorentz tradition, was not what Jordan had in mind, with his emphasis on the interaction of 3-dimensional waves. The first priority, then, was to find a plausible application of the Maxwellian view of electromagnetic interactions in the presence of continuous charge distributions, to the radically altered situation of the 1920's.

Schrödinger's electromagnetic interpretation of quantum mechanics has enjoyed wide documentation in the secondary literature; it is sufficient to recall that whereas it carries through perfectly adequately in the 1-particle case with an external interaction, the difficulties in the

n-particle case appeared insuperable (because of the 3n dimensionality of the configuration space, on which the wave functions were defined). Schrödinger had himself pointed out some of these difficulties, and in the process indicated the equations of motion that one would expect from a classical conception of the self-interaction of continuous charged matter fields: it was to these equations that Jordan now turned. They were:

$$\begin{aligned} &(-\hbar^2/2m)\ \Delta\ \phi + \frac{\hbar}{i}\frac{\partial}{\partial t}\ \phi - eV\phi = 0 \\ &(-\hbar^2/2m)\ \Delta\ \bar{\phi} - \frac{\hbar}{i}\frac{\partial}{\partial t}\ \bar{\phi} - eV\bar{\phi} = 0 \\ &\Delta V = 4\pi\ e\bar{\phi}\ \phi \end{aligned} \tag{3}$$

The Laplace equation, the third of Eq.(3) (in which e is the magnitude of the charge), follows from a theory of electrostatics for a continuous charge distribution with charge density $-e\bar{\phi}\phi(\mathbf{x},t)$; the first and second for e = 0 follow from the wave mechanics for a single free particle, which was interpreted by Schrödinger as the *classical* field equations for a continuous matter field of mass density $\bar{\phi}\phi(\mathbf{x},t)$. The relativistic analogue of these equations had recently been investigated by Gordan in some detail (Gordan [1926]; see (1.4.1)). Concerning this treatment, Schrödinger wrote:

> If we now ask ourselves whether this self-contained theory of the field - apart from the provisional neglect of the electronic spin - corresponds to reality in the way we had previously hoped for from such theories, the question must be answered *in the negative*. The examples worked out, particularly that of the H-atom, show in fact that we must *not* insert in the wave equation
>
> $$\left[\left(\partial/\partial x_i + i\ \frac{e}{\hbar}\ \varphi_i\right)^2 - m^2c^2/\hbar^2\ \right]\ \psi = 0 \tag{4}$$
>
> those potentials which result from the potential equations
>
> $$\Box\ \varphi_i = j_i \tag{5}$$
>
> with the 4-current
>
> $$j_i = -ie\ \psi(\partial/\partial x_i - ie\varphi_i)\bar{\psi}\ + c.c. \tag{6}$$
>
> On the contrary, we know that in the case of the H-atom we have to susbstitute the *given* potentials of the nucleus and of possible "external" electromagnetic fields for the φ_i and solve the equation for ψ (Schrödinger

[1927 p.135])[8].

We shall consider Schrödinger's critique of the matter-wave interpretation in (1.3.3) below. But the fundamental difference, between the system Eq.(3) and the wave mechanics for one particle, is precisely this: the Schrödinger equation is *always* linear.

Jordan was fully aware of this fundamental difference; in his collaboration with Klein "Zum Mehrkorproblem der Quantentheorie", submitted in October of 1927, referring to the system of equations considered by Gordan and Schrödinger, he wrote:

> ...we possess in the wave mechanics of the 1-particle problem a "classical model" which satisfies the requirements of the principle of relativity, and it is therefore obvious to attempt to see whether one can use this as a basis for the many-body problem, by subjecting the quantities which arise - the electromagnetic potential and Schrödinger wave function - to a "quantization". As a preparation for such an attempt we show in the present work that it is possible, from this point of view, in the special case where the finite propagation of the field distribution is neglected, to reach agreement with the wave equations in configuration space... (Jordan and Klein [1927 p.752]).

Shortly following he refers to the critique of Schrödinger above and claims that this problem is solved in the new theory, which was a canonical field quantization of the de Broglie field. The system Eq.(3) is obtained as the Euler-Lagrange equations from the Lagrangian density:

$$\mathcal{L} = -(\hbar^2/2m)\nabla\bar{\psi}\,.\,\nabla\psi + i\hbar(\bar{\psi}\,\partial\psi/\partial t - \partial\bar{\psi}/\partial t\,\psi) + eV\,\bar{\psi}\,\psi + (1/8\pi)\nabla(V-W).\nabla(V-W)$$

in which an external divergence-free potenial W is included. From this we deduce that $(i\hbar)\psi$ and $\bar{\psi}$ are canonically conjugate coordinates[9]; since $\partial V/\partial t$ does not appear in the Lagrangian , the canonical momentum corresponding to V is zero. Consequently $\dot{V}$ is not an independent canonical coordinate, and the variational equation generated by

[8] Page numbers refer to the English translation.

[9] Recall that the canonically conjugate coordinate π to a field ϕ is formally defined as $\pi = \partial\mathcal{L}/\partial\dot{\phi}$

variation in V is correspondingly a constraint (the third of Eq.(3)). Integrating this equation we obtain V as a function of the ψ's, that is:

$$V(\mathbf{x},t) = W(\mathbf{x}) - e \int G(\mathbf{x},\mathbf{x}')\bar{\psi}(\mathbf{x}',t)\psi(\mathbf{x}',t)\, d^3x'$$

The Green's function G is given by:

$$G(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|}$$

If one now introduces the expansion

$$\psi(\mathbf{x},t) = \sum_j b_j(t)u_j(\mathbf{x}) \tag{7}$$

where as before

$$(\hbar^2/2m)\Delta\, u_j = -E_j u_j$$

one obtains the Lagrangian $L = \int \mathcal{L}\, d^3x$ given by:

$$-\sum_{jk} H^1_{jk}\bar{b}_j b_k - \frac{e}{2}\sum^2_{jklm} H^2_{jk;lm}\bar{b}_j b_k \bar{b}_l b_m + \frac{i\hbar}{2}\sum_j \bar{b}_j \dot{b}_j - \bar{b}_j b_j \tag{8}$$

where

$$H^1_{jk} = \int \bar{u}_j\left[(-\hbar^2/2m)\Delta + W\right]u_k\, d^3x \tag{9}$$

$$H^2_{jk;lm} = \iint G(\mathbf{x},\mathbf{x}')\bar{u}_j(\mathbf{x})u_k(\mathbf{x}')\bar{u}_k(\mathbf{x})u_m(\mathbf{x}')\; d^3x d^3x'$$

and in the standard way[10] the Hamiltonian is obtained as:

$$\hat{H} = \sum_{jk} H^1_{jk}\bar{b}_j b_k + (e^2/2)\sum_{jklm} H^2_{jk;lm}\bar{b}_j b_k \bar{b}_l b_m \tag{10}$$

from which the equations of motion for the fields may be recovered. When e is zero Eq.(10) yields the Hamiltonian obtained by Dirac; alternatively, assuming the existence of functions $w_k(\mathbf{x})$ such that

$$\left[(\hbar^2/2m)\Delta - W\right]w_k = E'_k\, w_k$$

and expanding with respect to this set, (obtaining new expansion coefficients a_k), one finds:

$$\hat{H} = \sum_j E'_j \bar{a}_j a_j + (e^2/2)\sum_{jklm} H^2_{jk;lm}\bar{a}_j a_k \bar{a}_l a_m . \tag{11}$$

This is the expression which Jordan and Klein actually obtained. If one considers the solution manifold of the field equations, Eq.(3), as a Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3, d^3x)$ with inner product $\langle f,g\rangle_1 = \int \bar{f}g\, d^3x$ and $\langle .,. \rangle_2$ on $\mathcal{H}\otimes\mathcal{H}$, then Eq.(9) take the form:

$$H^1_{jk} = \langle u_j, H^1 u_k\rangle_1$$

$$H^2_{jk;lm} = \langle u_{jk}, H^2 u_{lm}\rangle_2$$

where $u_{jk} = u_j \otimes u_k$ span $\mathcal{H}\otimes H$; as we shall see, when the b's

[10] That is, $H = \int \mathcal{H}(\mathbf{x})d^3x$, $\mathcal{H} = \Sigma_i \pi_i \phi_i - \mathcal{L}$, where π_i, ϕ_i are canonically conjugate variables.

are assumed to satisfy the CCR's[11]:

$$[\hat{b}_j, \hat{b}^*_k] = \delta_{jk}, \quad [\hat{b}_j, \hat{b}_k] = [\hat{b}^*_j, \hat{b}^*_k] = 0 \qquad (12)$$

Eq.(9) defines the sum of the *canonical second quantizations of the 1-particle and 2-particle Hamiltonians* H^1 *and* H^2.

Jordan and Klein followed Dirac in introducing number and phase operators as canonically conjugate variables, in place of the b's (or rather the a's, since they used the basis w_k, the stationary states of the field including the external potential W), and assumed the Hamiltonian $\hat{H}$ acted on square intergrable functions of the occupation numbers, *i.e.* of the form:

$$\phi = \phi\,(n_1, n_2, \ldots, n_i, \ldots)$$

to obtain the Schrödinger equation:

$$i\hbar\, \partial\phi/\partial t = \sum_j E'_s N_s \phi +$$

$$\frac{e^2}{2} \sum_{jklm} H^2_{jk;lm} N_m^{1/2} (N_l - \delta_{lm})^{1/2} (N_k + 1 - \delta_{km} - \delta_{kl})^{1/2} (N_j + 1 + \delta_{jk} - \delta_{jm} - \delta_{jl})^{1/2}$$

$$\cdot\, e^{i(\theta_l + \theta_m - \theta_j - \theta_k)/\hbar}\, \phi \qquad (13)$$

By procedures exactly the same as those used by Dirac in proving the equivalence of method 2 with method 3, the authors proceeded to establish that Eq.(13) is equivalent to the Schrödinger equation for a system of $n = n_1 + n_2 + \ldots + n_i + \ldots$ bosons subject to Coulomb interactions[12], *i.e.* a Hamiltonian of the form[13]:

$$H = \oplus_{r=1}^{n} H^1_r + \oplus_{r \neq s=1}^{n} H^2_{rs}$$

where:

[11] In the sequel a function f considered as an operator is denoted $\hat{f}$; correspondingly the complex conjugate $\bar{f}$ is replaced by the adjoint operator $\hat{f}^*$. It was in this paper that Jordan first introduced the notation † for the adjoint of an operator. We shall always use the symbol * .

[12] I shall not discuss this equivalence proof. For a comprehensive review see Tomonaga [1966 p.315-326].

[13] I have slightly reformulated Jordan and Klein's treatment.

$$H^1_r = \mathbb{I}\otimes\mathbb{I}\otimes\ldots \quad \ldots \quad \mathbb{I}\otimes H^1\otimes\mathbb{I}\otimes\ldots \quad .. \quad .\otimes\mathbb{I}$$
$$\longleftarrow \text{r-1 terms} \longrightarrow \qquad \longleftarrow \text{n-r terms} \longrightarrow \qquad (14)$$

and

$$H^2_{rs} = \mathbb{I}\otimes\mathbb{I}\otimes\ldots \quad \ldots \quad \mathbb{I}\otimes\tilde{\mathbb{I}}\otimes\mathbb{I}\otimes\ldots \quad \otimes\mathbb{I}\otimes\tilde{\mathbb{I}}\otimes\mathbb{I}\otimes\ldots \quad \ldots\otimes\mathbb{I}$$
$$\longleftarrow \text{r-1 terms} \longrightarrow$$
$$\longleftarrow\!\!\longleftarrow \text{s-1 terms} \longrightarrow\!\!\longrightarrow$$

where $\sim\ \sim$ means that the correponding pair of operators is to be ommitted, to be replaced by the two-particle operator

$$\frac{e^2}{|\mathbf{x}-\mathbf{x}'|}: \quad L^2(\mathbb{R}^3,d^3x)\otimes L^2(\mathbb{R}^3,d^3x') \longrightarrow L^2(\mathbb{R}^3,d^3x)\otimes L^2(\mathbb{R}^3,d^3x') \quad (15)$$

The authors also wrote down the commutation relationships satisfied by the quantum fields, defined by Eq.(2), and obtained by appeal to the CCR's Eq.(12) and the (assumed) completeness relationship:

$$\sum_{jk}\bar{u}_j(\mathbf{x})u_k(\mathbf{x}') = \delta^3(\mathbf{x}-\mathbf{x}')$$

i.e.

$$[\hat{\psi}^*(\mathbf{x},t),\ (\hbar/i)\hat{\psi}(\mathbf{x}',t)] = i\hbar\,\delta^3(\mathbf{x}-\mathbf{x}') \quad (16)$$

this is the first time that commutation relationships for *spacetime quantum fields* were written down.

1.3.3. The field - many-particle equivalence: methods 2 and 3

In this short paper of Jordan and Klein the outlines of a general correspondence between any many-particle boson system with pair interactions, and a correponding non-linear quantum field theory, were spelt out. On a purely technical level (and particularly since the theory is in fact Galilean covariant) this achievement - only two years after Heisenberg's breakthrough - should be compared with what should be exactly the same theory, but with the Lorentz group in place of the Galilei group. No such equivalence exists; and after a half-century of intensive effort no comparable Hilbert space theory has been produced.

The equivalence that exists for the non-relativistic theory is of interest in its own right. But I want to keep in mind also the situtation for the relativistic theory - that there it *fails*. I believe this fact sets up a philosophical puzzle; the basic structure of quantum theory does not appear to make any critical reference to the spacetime

group, and there is no obvious reason why one should not simply take over the field quantization above beginning from a Lagrangian which is Lorentz invariant rather than Galilean invariant. Of course physically one knows that in the relativistic theory pair production and annihilation is possible, so that the equivalence proof cannot go through in this form, since the total particle number is not in general conserved. But the idea (or necessity) of pair production and annihilation and the attendant concept of antiparticle follows from the field quantization, therefore once again from the mere change in symmetry group. One wants to know why[14].

One of the things which makes the non-relativistic equivalence possible is the fact that the Lagrangian, and therefore the Hamiltonian, is bilinear in the fields; the difficulties with the QED equivalence proof stemmed from the fact that the interaction Hamiltonian is linear in the electromagnetic potential, which therefore induces transitions differing in photon number. If, in the foregoing, the external c-number potential W is considered a function of the electromagnetic field variables (and the free-field Hamiltonian included) one will have exactly the same problem, in defining an equivalent particle theory for the radiation field. This problem appears so intractable that we shall consider it insoluble; there simply is no equivalence when individual creation and annihilation processes take place, that is when the Hamiltonian is linear in a quantum field. This is typical of a non-linear theory; throughout this thesis we shall be concerned with the equivalence only in the *linear* case, which excludes such processes.

But for linear theories the equivalence is by no means

[14] The usual heuristic argument, from the existence of kinematically permitted motions of negative energy to the existence of pair creation and annihilation processes, is clearly along the right lines. The difficulty is to make the argument precise. This will be a dominant theme in Part 3.

guaranteed; another factor, equally important, is that the particle interpretation of the fields is so simple; the fields *are* the creation and annihilation operators, not linear combinations of them, and the interaction Hamiltonian of the form[15]

$$e\int \hat{\psi}^*(\mathbf{x})\hat{V}(\mathbf{x})\hat{\psi}(\mathbf{x})d^3x + (1/8\pi)\int \nabla\,(V(\mathbf{x})-W(\mathbf{x})).\nabla(V(\mathbf{x})-W(\mathbf{x}))\,d^3x$$
$$= (e^2/2)\iint G(\mathbf{x},\mathbf{x}')\hat{\psi}^*(\mathbf{x})\hat{\psi}^*(\mathbf{x}')\hat{\psi}(\mathbf{x}')\hat{\psi}(\mathbf{x})d^3xd^3x' = \hat{H}_I \qquad (17)$$

clearly induces no change in total particle number. We have, in fact, normal-ordered this expression (placed all annihilation operators to the right of all creation operators), a point that Jordan and Klein discussed in some detail. Without this normal-ordering one obtains (we omit the factor $(e^2/2)$, by which all terms should be multiplied):

$$\hat{H}_I' = \iint G(\mathbf{x},\mathbf{x}')\hat{\psi}^*(\mathbf{x})\hat{\psi}(\mathbf{x})\hat{\psi}^*(\mathbf{x}')\hat{\psi}(\mathbf{x}')\,d^3xd^3x'$$
$$= \hat{H}_I + \int G(\mathbf{x},\mathbf{x})\hat{\psi}^*(\mathbf{x})\hat{\psi}(\mathbf{x})d^3x = \hat{H}_I + \hat{H}_S \qquad (18)$$

where $\hat{H}_S$, the self-interaction energy, is clearly singular. Yet these formula precisely parallel the classical interaction energy for a system of point masses (indexed by i,j):

$$\sum_{i,j} G(\mathbf{x}_i,\mathbf{x}_j) = \sum_{i\neq j} G(\mathbf{x}_i,\mathbf{x}_j) + \sum_i G(\mathbf{x}_i,\mathbf{x}_i) \qquad (19)$$

The similarity with Eq.(18) is all the more evident when we write terms of the form $\hat{\psi}^*(\mathbf{x})\hat{\psi}(\mathbf{x})$ simply as $\hat{N}(\mathbf{x})$, the number density operator:

$$\hat{H}_I' = \iint G(\mathbf{x},\mathbf{x}')\hat{N}(\mathbf{x})\hat{N}(\mathbf{x}')\,d^3xd^3x' = \hat{H}_I + \int G(\mathbf{x},\mathbf{x})\hat{N}(\mathbf{x})\,d^3x \qquad (20)$$

That is, in field theory we simply take the continuum limit[16] of Eq.(19), with $N(\mathbf{x})d^3x$ particles in the volume element d^3x, except for the first term on the RHS of Eq.(19)

[15] In the second term we perform a partial intergration and use the fact that $\Delta G(\mathbf{x},\mathbf{x}') = 4\pi\delta^3(\mathbf{x}-\mathbf{x}')$.

[16] One can write Eq.(19) in continuum terms, in the classical theory, by using delta function charge distributions:

$$\sum_{i,j} G(\mathbf{x}_i,\mathbf{x}_j) = \iint G(\mathbf{x},\mathbf{x}')N(\mathbf{x})N(\mathbf{x}')d^3xd^3x'$$

where $N(\mathbf{x}) = \sum_i \delta^3(\mathbf{x}-\mathbf{x}_i)$. Distributions have widespread applications in classical mechanics (see, e.g. Kecs and Teodorescu [1974]); contrary to Redhead [1983] I do not, however, consider such a formulation of classical mechanics a classical field theory, because the dynamics cannot be expressed in local form, which I consider the fundamental distinctive property of field theory.

(Eq.(17)) which cannot be expressed in this way:

> The result for the interaction energy which we obtained above is therefore in fact the same as that which we must expect by analogy with the classical theory of point masses. Remarkably, the noncommutative multiplication of quantum mechanics makes it possible to express the difference between a double and a single volume integral in (the continuum limit of) Eq.(19) by a single double volume integral; thereby it is possible in the quantum mechanics, in a unified manner, to express the difference between the action of the wave field (Eigenfeld) of an electron on itself and the action of the external wave field - a distinction that appears to be very satisfactory, and formulated exactly only with difficulty, in the classical theory. (Jordan and Klein [1927 p.762]).

Schodinger's electromagnetic interpretation

This feature of the field - particle correspondence has been remarked upon elsewhere[17], but not its connection with the Schrödinger electromagnetic interpretation - in particular with his critique, the most complete account that we possess of this interpretation, contained in his paper "the energy-momentum theorem for material waves." The following extract is the continuation of the passage quoted above; ψ is the electron wave function, and φ the potential which satisfies the d'Alembertian equation, Eq.(3):

> On the contrary, we know that in the case of the H-atom we must substitute the *given* potentials of the nucleus and the possible "external" electromagnetic fields for the φ in Eq.(6), and solve the equation for ψ. The distribution of current produced by this ψ is then calculated from Eq.(6) and from the distribution the potentials produced by it are found by Eq.(5). By adding the latter to the potentials given in advance, we obtain those potentials which define the external action of the atom as a whole. We thus obtain (with a suitable normalization of ψ,...) the *neutralisation* of the nuclear charge at greater distances on the one hand, and on the other hand the *radiation.* With reference to the attempt, which it would now be natural to make, to substitute these newly found potentials in Eq.(6), and thus to calculate a "second approximation", it is to be remarked that we must not on any account proceed in this way with the *neutralisation potential*, as it would *completely* alter the value of the terms, and hence would make more

[17] Schiff [1968 p.527], Schweber [1961 p.142].

stages of approximation necessary. These, even if the process converges at all, certainly do *not* lead to the correct hydrogen terms, much less (in the case of nuclear charge 2) to the helium *atom* terms. On the contrary, we should very probably obtain the required *radiation correction* by dealing with the *radiation potentials* in the way described, if we suppose that *one* proper vibration is *strongly* excited but all the others only very feebly.

Hence there is something which intrudes into the self-contained system of field equations in a peculiar way. This is not yet fully intelligible at present, but it must be considered in connection with the following two facts:
(i) The exchange of energy and momentum between the electromagnetic field and "matter" does *not* in reality take place continuously as the expression in terms of the field would lead us to believe.
(ii) In Lorentz's theory we also have to substitute in the first instance only the fields of the *other* electrons in the equations of motion of the single electron, and not its own individual field. The reaction of the latter has *already* been almost entirely taken account of as electromagnetic *mass, in setting up the equations of motion*. The corresponding terms in Eq.(6) is the term $(mc/\hbar)^2$. The reaction of radiation results in a second approximation from the reaction of the electron's own field in Lorentz's theory also.

The question whether the solution of the difficulty is really to be found only in the purely *statistical* interpretation of the field theory[18] which has been proposed in several quarters must for the present be left unsettled. Personally I no longer regard this interpretation as a finally satisfactory one, even if it proves useful in practise. To me it seems to mean a renunciation, much too fundamental in principle, of all attempt to understand the individual process.

A *brighter* side of the difficulty in question deserves to be mentioned. By interrupting the completeness of the system of field equations in her actual behaviour, nature accomodates herself to our mathematical powers to an astonishing extent. Even the theory of the hydrogen atom would become immeasurably complicated from the mathematical point of view, if the φ did not stand for *given* potential values in Eq.(6), but if instead we had to add to them those which are to be calculated by means of Eqs.(5),(6) from ψ, which is itself unknown. (Schrödinger [1927 p.135-6].

I have quoted from this paper at length because I do not

[18] Here he means the field theory understood as a 1-particle wave mechanics, not as a quantum field theory.

believe sufficient justice has been done[19] to the Schrödinger electromagnetic interpretation, applied to the *quantized* de Broglie field. I shall systematize the discussion by reference to this passage[20], in particular the following points:

A: the self-interaction of the field (Schrödinger's point (i)).
B: the fact that one does not, in the wave mechanics, substitute the potential as a functional of the wave function ψ.
C: the "brighter side" of the difficulty: nature's accomodation to our mathematical powers.
D: Schrödinger's dissatisfaction with the statistical interpretation, because it forgoes "all attempt to understand the individual process".
E: non-continuity in the exchange of energy between field and particle (Schrödinger's point (i)).

A is the easiest to deal with. As we have just seen, quantizing the field indeed includes the self-interaction in question, which - in the non-relativistic theory - has nothing to do with the mass. So this difficulty in the matter wave interpretation disappears (the analogue of B, but for the electron's own field).

B has its source in the fact that wave mechanics has the

[19] Schrödinger never considered the quantized de Broglie field in the general context of the interpretation of NRQM; neither have the numerous historians and philosophers of quantum physics who have discussed the Schrodinger electrogmagnetic interpretation, e.g. Jammer [1966], [1974] Scott [1967], Mackinnon [1976], Wessels [1979], Hanle [1977], Dorling [1987], Rohrlich [1985].

[20] With the possible exception of (ii), nothing in Schrödinger's critique hinges on the relativity group, and many problems attend its exegesis in the relativistic case (Part 3 is largely concerned with just these problems). The heuristic basis of the Schrödinger electromagnetic interpretation applies very simply to the non-relativistic theory; as indeed do many of his formal arguments.

fundamental structure of Hamiltonian particle mechanics; the potential V is not fixed in advance, only its functional dependence on the canonical variables. Solving the Hamiltonian equations then determines the canonical variables as functions of the time, and if these solutions are substituted back into the potential function this function will obey the Laplace equation with the particle positions (as functions of time) as sources. There is no question of an iterative procedure.

Explicitly, suppose we have a NRQM description of an n-particle system, described by the density matrix $\rho(t)$, with interaction Hamiltonian

$$V(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n) = (1/2)\sum_{\mathbf{x}_i\neq\mathbf{x}_j}\frac{q_iq_j}{|\mathbf{x}_i-\mathbf{x}_j|}$$

in which q_i, q_j are the charges of particles i and j. Define the function $\langle V\rangle_\rho(\mathbf{x},t) = \mathrm{Tr}(V(\mathbf{x})\rho(t))$ where

$$V(\mathbf{x}) = \sum_j \frac{q_j}{|\mathbf{x}-\mathbf{x}_j|}$$

And consider this function when ρ is a pure state, defined by the vector ψ in

$$\mathcal{H}^n = L^2(\mathbb{R}^3,d^3x_1)\otimes L^2(\mathbb{R}^3,d^3x_2)\otimes\ldots\otimes L^2(\mathbb{R}^3,d^3x_n)$$

i.e. with $\Psi(\mathbf{x}_1,\mathbf{x}_2,..,\mathbf{x}_n;t) = e^{-iHt/\hbar}\psi$:

$$\langle V\rangle(\mathbf{x},t) = \int \overline{\Psi(\mathbf{x}_1\ldots\mathbf{x}_n;t)}\sum_j\frac{q_j}{|\mathbf{x}-\mathbf{x}_j|}\Psi(\mathbf{x}_1\ldots\mathbf{x}_n;t)\,d^3x_1\ldots d^3x_n$$

It is then apparent that:

$$\Delta\langle V\rangle(\mathbf{x},t) = \int \overline{\Psi(\mathbf{x}_1\ldots\mathbf{x}_n)}\sum_j 4\pi q_j\delta^3(\mathbf{x}-\mathbf{x}_j)\,\Psi(\mathbf{x}_1\ldots\mathbf{x}_n;t)d^3x_1\ldots d^3x_n$$

in particular when ψ is a product state $\psi = \psi_1\otimes\ldots\otimes\psi_n$ one has:

$$\Delta\langle V\rangle(\mathbf{x},0) = 4\pi q_1\overline{\Psi_1(\mathbf{x}_1,0)}\Psi_1(\mathbf{x}_1,0) + ..+ 4\pi q_n\overline{\Psi_n(\mathbf{x}_n,0)}\Psi_n(\mathbf{x}_n,0)$$

Of course the state will evolve into a state which is *not* a product state, which is why we consider $\Delta\langle V\rangle(\mathbf{x},t)$ at $t=0$; the general case is not difficult to establish, and the agreement with the third of Eq.(3) still follows.

The function $\langle V\rangle(\mathbf{x},t)$ does not play any rôle in NRQM, for $\mathbf{x}$ is not a canonical variable. We can provide a (heuristic) canonical meaning for this function, however, by considering a system of n+1 particles, the additional particle having sufficiently small charge such that its influence on the

n-particle system may be neglected; that is, if the n+1 particle state is of the form $\phi\otimes\psi$, with ψ as above, the time-dependent wave function Ψ remains (approximately) of this form with projection ψ onto $\mathcal{H}^n$. This approximation is valid when the term in square brackets in the n+1 Schrödinger equation vanishes:

$$(-\hbar^2/2m)(\Delta\phi)\otimes\Psi - \sum_{j=1}^{n}\frac{q\,q_j}{|\mathbf{x}-\mathbf{x}_j|}\,\phi\otimes\Psi - i\hbar(\partial\phi/\partial t)\otimes\Psi =$$

$$\phi\otimes\left[\sum_{j=1}^{n}(\hbar^2/2m_j)\Delta_j\Psi + (1/2)\sum_{i\neq j}\frac{q_i q_j}{|\mathbf{x}_i-\mathbf{x}_j|}\,\Psi + i\hbar\partial\Psi/\partial t\right].$$

(This is no different from the usual definition of a field, in the framework of classical mechanics[21], as the field at the position of a test charge q, $q \longrightarrow 0$, of vanishingly small influence on the field.)

Schrödinger's critique B is concerned rather with mathematical method than with the "real" relationship between the potential function and its sources; as such it is eliminated in the quantized de Broglie theory, where, as a question of mathematical method, one does indeed explicitly substitute the integral expression for the potential, in terms of the $\hat{\psi}$'s, *before* solving the dynamical problem. That is consistent with the foregoing, however, because now the $\hat{\psi}$'s *are* dyanical variables. This is true classically as well, which brings us on to C. Classically, as a matter of mathematical technique, one would indeed be led to an iteration scheme in the ψ's; as a QFT, however, not only is it possible to express the (renormalized) interaction energy "by a single double volume integral" (in Jordan's words), but also the non-linear system is formulated as a linear differential equation (the Schrödinger equation for the quantized de Broglie field).

So much is true for interacting RQFT as well; and this is a remarkable feature of quantization. In the non-relativistic case, however, a further simplification is possible, by

[21] Or indeed on the general philosophy, that theoretical terms be defined operationally.

exploiting the field - many-particle equivalence, and this single linear equation may be written as a system of 3n linear differential equations.

In D Schrödinger expresses one of the central grounds for dissatisfaction with NRQM, that it does not, on the face of it, describe the individual system. I believe the latter is a fallacy, but I wish to point out that for those who are so dissatisfied, the situation appears somewhat different with regard to the de Broglie field. One no longer begins with a 1-particle system, and then interprets predictions of the theory in terms of the statistical properties of an ensemble of particles. On the contrary, interference phenomenology (fringes etc.) always involve an ensemble of large but indeterminate particle number and the observed *local* phenomenology is directly given by the expectation values of local bilinear products in the quantum fields (see below).

E is one form of the measurement problem; in brief, as I elaborate in (2.3.5) and in detail in Section 3.5, there is an apparent "collapse of the wave function" which only occurs in a certain class of specific interactions (essentially when a microscopic system is coupled to a system of large number of degrees of freedom), to be described as any other kind of interaction, with the consequence - following from the canonical structure of quantum theory alone - that quantum probabilities become subject to an ignorance interpretation of probability. There is no collapse of the wave-function; for such interactions and for such systems it evolves into a state arbitrarily close to an incoherent mixture over the possible macroscopic outcomes.

I wish to conclude that all the heuristic content of the Schrödinger electromagnetic interpretation is taken over directly by considering the wave mechanics as a fragmentary expression of the behaviour of the quantized de Broglie field. The fundamental residual difficulty is purely the problem: in what sense can operator-valued functions on

spacetime be considered (the mathematical description of) physical entities? I consider this problem is no more and no less than the problem: how can we formulate intuitions about putative physical entities so-described? I do not pretend that this is easy - but it is surely not insurmountable. Further, I believe one can advance good reasons why one *should* be led to this sort of description of physical systems. That is the business of Part 2, in particular (2.3.5) and (2.5.3),

The Schrödinger construction

To substantiate the claim that the heuristics of the electromagnetic interpretation are immediately applicable to the quantized de Broglie field, I will consider one last aspect of the field - many-particle correspondence. The most frequent objection to the electromagnetic interpretation focused on the dimensionality of the domain of the wave-function for many-particle systems; Schrödinger had frequently described the general interpretation, in which the modulus square of the wave-function is taken to define the charge density in 3-dimensional space, in contexts in which the function was in fact defined on a 3n-dimensional Euclidean space. On the one occasion in which he explicitly addressed the problem, he wrote:

> According to the *heuristic hypothesis* on the electrodynamical significance of the field scalar ψ, the present quantity ...(of the form $\overline{\psi(x)}\psi(x)$) ...represents the electrical density as a function of the space co-ordinates, and the time, *if* **x** *stands for only 3 space coordinates, i.e* if we are dealing with problem of *one* electron. By a natural generalization of this hypothesis, we regard the following as representing in the general case the density of the electricity, which is "associated" with *one* of the particles of classical mechanics, or which "originates in it", or which "corresponds to it in wave mechanics": the *integral* of $\overline{\psi}\psi$ taken over all those coordinates of the system, which in classical mechanics fix the position of the *rest of the* particles.... the resulting density of charge at any point in space is then represented by the *sum* of such integrals taken over all the particles. (Schrödinger [1927 p.109]).

We shall refer to this proposal, for the interpretation of

the (configuration space) state of an ensemble as the *Schrödinger construction.* Evidently this is just what we obtain in Eq.(20); however it is clear the construction emphasises in an essential way the particle interpretation of the state, that in some sense one is driven to a dualistic interpretation which refers both to the wave function *qua* field, and the wave function *qua* state of a many-particle ensemble. The construction also appears contrived from a mathematical point of view, although it (or something very similar) may be derived from a systematic application of the Born interpretation to projection operators associated with a particle ensemble.

We can formalize this interpretation by considering the Hilbert space $\mathcal{H}^n$ (the tensor product of L^2 spaces on $\mathbb{R}^3$ as in the foregoing), and defining the multiplicative bilinear form[22] on this space given by:

$$\delta_{\mathbf{x}}: f(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) \longrightarrow q \sum_{j=1}^{n} \delta^3(\mathbf{x}-\mathbf{x}_j) f(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) \qquad (21)$$

The Schrödinger construction can be defined through the simple prescription: the charge density at the point $\mathbf{x}$ for a system of n particles all of the same charge[23] is given by the expectation value of $\delta_{\mathbf{x}}$. For a product state this result is trivial. For a state of the form

$$\psi = \sum_{j_1, j_2, \ldots j_n} C_{j_1 j_2 \ldots j_n} \psi_{j_1 j_2 \ldots j_n}$$

in which $\psi_{j_1 j_2 \ldots}$ is a normalized product state of the form $\phi_{j_1} \otimes \phi_{j_2} \otimes \ldots \otimes \phi_{j_n}$, where particle 1 is in state $j_1, \ldots,$ particle n in state j_n, and the coefficients C are normalized to

$$\sum_{j_1 j_2 \ldots j_n} |C_{j_1 \ldots j_n}|^2 = 1 ,$$

one obtains:

[22] obviously $\delta_{\mathbf{x}}$ does not make sense as an operator on $\mathcal{H}^n$.

[23] It is easy to accomodate a system of different charges; we shall not need this, however.

$$\langle\psi,\delta_{\mathbf{x}}\psi\rangle = q\sum_{r,s=1}^{\infty}\overline{\phi_r(\mathbf{x})}\phi_s(\mathbf{x})\ .$$

$$\sum_{k,l=1}^{n}\ \sum_{j_1..\hat{j}_k,..\hat{j}_l,..}\overline{C_{j_1,..j_{k-1}rj_{k+1},..,j_n}}\,C_{j_1..j_{l-1}sj_{l+1},..j_n} + \text{c.c.}$$

in which r runs over the 1-particle orthonormal basis and $\hat{j}_k$ means that this term is dele-ted in the summation. Our experience with expressions of this form suggests that when C is a symmetric function of the j's we can find a simple expression in the occupation number representation. But we know well enough what we must end up with; it is the expectation value of the number density operator. It is a simple matter to calculate that for ψ of the above form in $\mathcal{H}^s$, which we may write:

$$\psi = \sum_{n_1,..,n_r,..} C_{n_1n_2...n_r...}\ \phi_{n_1...n_r...}$$

in which the n_r's are constrained to sum to n, and r still runs over the orthonormal 1-particle basis ϕ_r, then:

$$\langle\psi,\hat{\psi}^*(\mathbf{x})\hat{\psi}(\mathbf{x})\psi\rangle = \sum_{n_1...n_r..}\frac{1}{2}\sum_{r,s}n_r^{1/2}(n_s+1-\delta_{rs})^{1/2}\ . \tag{22}$$

$$\overline{C_{n_1..n_r-1..n_s+1..}}\,C_{n_1n_2..}\,\overline{\phi_s(\mathbf{x})}\phi_r(\mathbf{x}) + \text{c.c.}$$

Both of these problems are therefore eliminated in NRQFT; the Schrödinger construction appears naturally, as the expectation value of the number density operator, and one need not refer to the particle interpretation of the state explicitly.

Canonical second quantization: heuristics

As Jordan was to point out in a paper immediately following his collaboration with Klein (Jordan [1927c]), entitled "Uber Wellen und Korpuskeln in der Quantenmechanik" - a paper concerned almost entirely with the application of his transformation theory to the occupation number formalism - in a formal sense any one particle operator A may be second quantized as the operator[24]:

[24] There is no general rigorous proof of the this result, and each individual case must be separately analysed. For the operators of physical interest it is true nevertheless. The

$$d\Gamma(A) = \int \hat{\psi}^*(\mathbf{x})A(\mathbf{x})\hat{\psi}(\mathbf{x})\ d^3x. \qquad (23)$$

Here we have used standard notation on the LHS; $A(\mathbf{x})$ on the RHS is the configuration space representation of A, a c-number polynomial in $\mathbf{x}$ and derivatives in $\mathbf{x}$. Formally, then, for $\delta_{\mathbf{x}}$ the 1-particle version of the bilinear form introduced in Eq.(21), we have that:

$$d\Gamma(\delta_{\mathbf{x}}) = \hat{\psi}^*(\mathbf{x})\hat{\psi}(\mathbf{x}). \qquad (24)$$

Any 2-particle operator A is second quantized as:

$$d\Gamma(A) = \int \hat{\psi}^*(\mathbf{x})\hat{\psi}^*(\mathbf{y})A(\mathbf{x},\mathbf{y})\hat{\psi}(\mathbf{x})\hat{\psi}(\mathbf{y})\ d^3x d^3y$$

and so on. More generally, if $\hat{a}(u_k)$ is the annihilation operator for a particle in the state u_k (*i.e.* what we denoted above by $\hat{a}_k$), so that figuratively and condsidering $\delta_{\mathbf{x}}$ as the 1-particle improper position eigenstate with generalized eigenvalue $\mathbf{x}$, then $\hat{\psi}(\mathbf{x}) = \hat{a}(\delta_x)$, and can write Eq.(23) as:

$$d\Gamma(A) = \int \hat{a}^*(\delta_x)\langle\delta_{\mathbf{x}}, A\delta_{\mathbf{x}}\rangle\hat{a}(\delta_{\mathbf{x}})\ d^3x.$$

$$= \int \hat{a}^*(\delta_{\mathbf{x}})\delta^3(\mathbf{x}-\mathbf{z})A(\mathbf{z})\delta^3(\mathbf{y}-\mathbf{z})\hat{a}(\delta_{\mathbf{y}})\ d^3x d^3z d^3y$$

One can see what must happen in the general case; choosing instead a (genuine) orthonormal basis u_k we have:

$$d\Gamma(A) = \sum_{j,k} \hat{a}^*(u_k)\langle u_k, Au_j\rangle\hat{a}(u_j) \qquad (25)$$

which has a simple interpretation in terms of its action on $\mathcal{H}^n$: first it multiplies a separable state by $\langle u_k, Au_j\rangle$, correponding to the transition of a particle in the state u_j to a state u_k under the influence of the 1-particle operator A, and this is multiplied by the amplitude with which a given particle is in the state u_j. The state itself is replaced by the same state, but in which the particle is no longer in the state u_j but in the state u_k instead - and this process is then repeated for all the particles present, in such a way as to preserve symmetry and normalization.

The process is even more intuitively obvious for the second quantization of a 2-particle operator, given by:

$$d\Gamma(A) = \sum_{j,k,l,m} \hat{a}(u_j)\hat{a}(u_k)\langle u_{jk}, Au_{lm}\rangle_2\hat{a}(u_l)\hat{a}(u_m) \qquad (26)$$

point fields $\psi(\mathbf{x})$ must be considered bilinear forms, and not operators.

Acting on the state, particles in the states u_l and u_m are replaced by particles in the states u_j, u_k; the new state is multiplied by normalization factors and by the transition amplitude $\langle u_{jk}, Au_{lm} \rangle_2$ for the operator A to induce transitions from the one 2-particle system to the other (here $\langle .,. \rangle_2$ is the natural 2-particle inner product introduced in (1.3.2)). The process is then repeated for all particle pairs, so as to preserve symmetry and normalization; we refer to Fig. 1.3 for a visual representation.

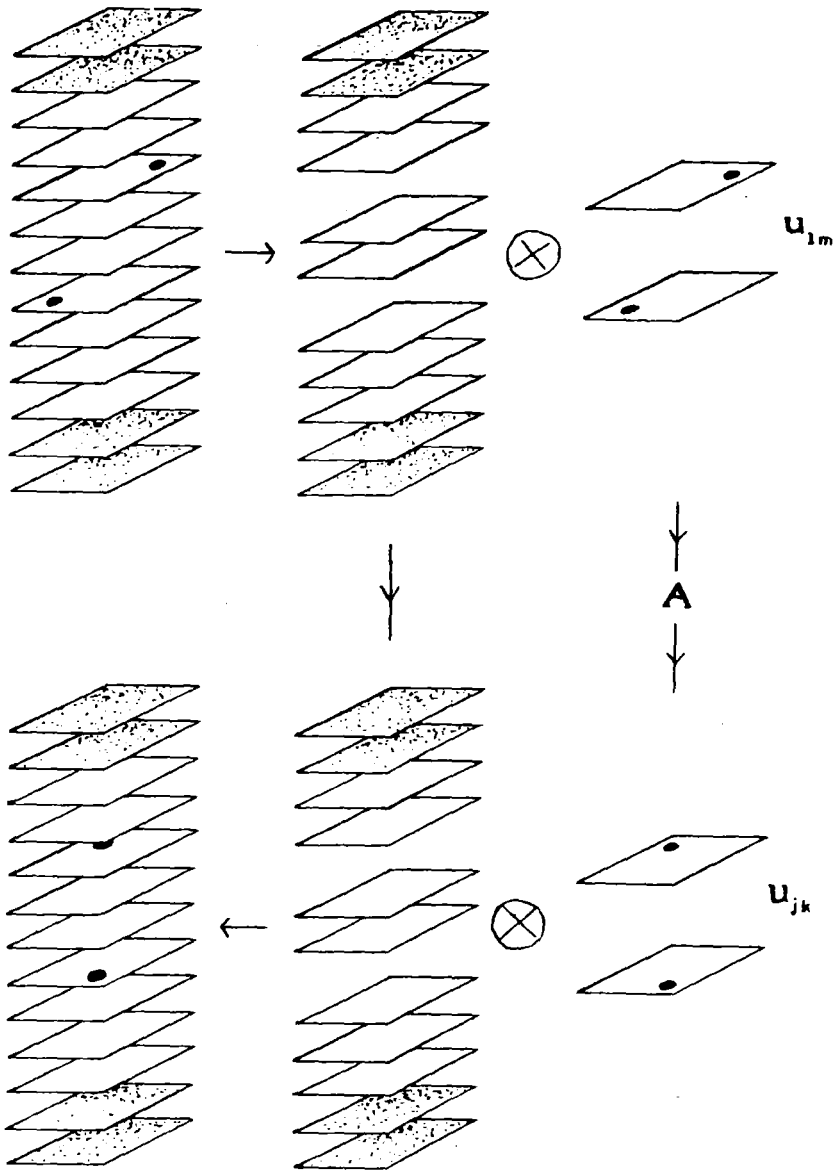


Fig. 1.3

The local correspondence

We conclude by emphasising that the electromagnetic interpretation applies specifically to the configuration space representation; in this representation one has what I shall call *the local correspondence of the field and 1-particle theory*, in the sense that any formal (configuration space) expression in the 1-particle theory may be directly written in second quantized form, simply by everywhere replacing the 1-particle wave function by the corresponding quantum field (as an operator-valued function on spacetime). Some examples:

1-particle expressions	**q.f.t. expressions**
$\int \overline{\psi(\mathbf{x})}\psi(\mathbf{x})d^3x$	$\int \hat{\psi}^*(\mathbf{x})\hat{\psi}(\mathbf{x})d^3x$
normalization	total number operator
$\overline{\psi(\mathbf{x})}\psi(\mathbf{x})$	$\hat{\psi}^*(\mathbf{x})\hat{\psi}(\mathbf{x})$
probability density	number density operator
$\int \overline{\psi(\mathbf{x})}(-i\hbar\nabla)\psi(\mathbf{x})\ d^3x$	$\int \hat{\psi}^*(\mathbf{x})(-i\hbar\nabla)\hat{\psi}(\mathbf{x})\ d^3x$
expectation value of momentum	total momentum operator
$\overline{\psi(\mathbf{x})}(-i\hbar\nabla)\psi(\mathbf{x})$	$\hat{\psi}^*(\mathbf{x})(-i\hbar\nabla)\hat{\psi}(\mathbf{x})$
momementum probability density.	momentum density flux operator
$\overline{\psi(\mathbf{x})}(-\hbar^2\Delta/2m)\psi(\mathbf{x})$	$\hat{\mathcal{H}}(\mathbf{x}) = \hat{\psi}^*(\mathbf{x})(-\hbar^2\Delta/2m)\hat{\psi}(\mathbf{x})$
energy probability density	Hamiltonian density operator

The essential point is that under the local correspondence *the electromagnetic interpretation of Schrödinger applies for arbitrary particle number to the associated quantum field expressions* (associated, that is, by the local correspondence). The expectation values of the associated field expressions yield the *statistical distributions of spacetime events*. Whenever "wave-like" phenomenology is empirically produced - typically interference fringes - one

must actually generate a particle ensemble in the laboratory; the expectation values of the appropriate local field quantities describe just this phenomenology.

One might also express the situation as follows: the field - many-particle equivalence says essentially no more than that there exists a *particle interpretation* for the QFT. But if there exists a local correspondence as well, then the "wave-like" properties of the 1-particle theory may be directly interpreted in terms of (expectation values of) field densities; in this situation the field - many-particle equivalence bears directly on the wave-particle duality.

From the point of view of RQFT it must be emphasised that the local correspondence depends critically on the configuration space representation, and the identity of the space integrals over the fields with the configuration space inner product, as well as the particle interpretation of the fields as creation and annihilation operators (rather than linear combinations of these). There is no straightforward local correspondence in RQFT. In Part 3 we shall see why that is so, and eventually construct a local correspondence for the relativistic theory as well.

We conclude this section with a short summary of the canonical second quantization, as developed by Fock [1932] and Cook [1953]. The generalization to fermion systems was made by Jordan [1927b] and Jordan and Wigner [1928]; the detailed history of these developments is of limited interest, and we do not pursue it.

1.3.4. Canonical second quantization[25].

We denote the 1-particle Hilbert space by $\mathcal{H}$, and define $\mathcal{H}^n = \mathcal{H}\otimes\mathcal{H}\otimes\ldots\otimes\mathcal{H}$ (n factors in all). We define the symmetric subspace of $\mathcal{H}^n$ explicitly as follows: to every permutation π in the permutation group Π_n of degree n there corresponds a unitary operator U_π on $\mathcal{H}^n$ uniquely defined as the bounded linear extension of the operator

$$U_\pi:\ u_1\otimes u_2\otimes\ldots\otimes u_n = u_{\pi(1)}\otimes u_{\pi(2)}\otimes\ldots\otimes u_{\pi(n)}$$

where $\{u_i\}$ is any orthonormal basis for $\mathcal{H}$. The ring generated by $\{U_\pi\}$ is n! dimensional. We define the **n-particle symmetrizer**:

$$S_n = (n!)^{-1}\sum_{\pi\in\Pi_n} U_\pi$$

and the **n-particle antisymmetrizer**:

$$A_n = (n!)^{-1}\sum_{\pi\in\Pi_n} \mathrm{sign}(\pi)\ U_\pi$$

where $\mathrm{sign}(\pi) = \pm 1$ depending on whether π is even or odd. The invariant subspaces of $\mathcal{H}^n$ under S_n, respectively A_n, define the **symmetric subspace**, respectively **antisymmetric subspace**, of $\mathcal{H}$. These are spanned by the totally symmeteric, respectively antisymmetric states in $\mathcal{H}^n$. We denote these subspaces by $\mathcal{H}^n_S$, $\mathcal{H}^n_A$ respectively.

The distinctive new construction that at once frees the theory from reference to any particular particle number, and also leaves open the possibility that the evolution will result in a change in particle number, is that we take the direct sum (for all n) of the $\mathcal{H}^n$'s. That is, we define:

$$\mathfrak{F}(\mathcal{H}) = \oplus_{n=0}^{\infty}\mathcal{H}^n \qquad (27)$$

where $\mathcal{H}^0 = \mathbb{C}$. $\mathfrak{F}(\mathcal{H})$ is called **Fock space**. Its symmetric subspace:

$$\mathfrak{F}_S(\mathcal{H}) = \oplus_{n=0}^{\infty} S_n\mathcal{H}^n$$

[25] This subsection is a summary; for all proofs we refer to Cook [1953], Reed and Simon [1975]. Most of the results are already familiar from the forgoing, except for Eq.(48), the precise formal definition of the point creation and annihilation operators (Eq.(48)), and the definition of the Segal field and Weyl algebra (Eqs.(40)-(44)), to which we particularly draw attention. The equivalent theory of fermionic second quantization is also summarized.

is called the **symmetric Fock space**. We note the theorem[26]:

$$\mathfrak{F}_S(\mathcal{H}_1 \oplus \mathcal{H}_2) = \mathfrak{F}_S(\mathcal{H}_1) \otimes \mathfrak{F}_S(\mathcal{H}_2) \tag{28}$$

for an arbitrary pair of Hilbert spaces, $\mathcal{H}_1$, $\mathcal{H}_2$. This result has prompted the notation: $\mathfrak{F}_S(\mathcal{H}) = e^{\mathcal{H}}$, a useful mnemonic.

A vector $\psi = \eta^0 \oplus \eta^1 \oplus \eta^2 \oplus \ldots \oplus \eta^n \oplus \ldots$ where each $\eta^n \in \mathcal{H}_S^n$, such that $\eta^n = 0$ for all but finitely many n is called a **finite particle vector**. We denote the set of all such $F_S^0 \subseteq \mathfrak{F}_S(\mathcal{H})$.

The **canonical second quantized** 1-particle operators are defined as previously, except that we now must allow for arbitrary n. Let A be any densely defined linear transformation on $\mathcal{H}$ with domain D, and let $D_A = \{\psi \in F_S^0;\ \eta^n \in \otimes_{i=1}^n D \text{ for each } n\}$. We define $d\Gamma(A)$ on $D_A \cap \mathcal{H}_S^n$ as:

$$A \otimes \mathbb{I} \otimes \ldots \otimes \mathbb{I} \ \oplus\ \mathbb{I} \otimes A \otimes \mathbb{I} \otimes \ldots \otimes \mathbb{I} \ \oplus\ \ldots . \oplus \mathbb{I} \otimes \ldots \otimes \mathbb{I} \otimes A \tag{29}$$

This operator is closable and we take its closure, also denoted $d\Gamma(A)$. $d\Gamma$ satisfies the following properties:

(i) **linearity**: $d\Gamma(\alpha A + \beta B) = \alpha d\Gamma(A) + \beta d\Gamma(B)$, $\alpha, \beta \in \mathbb{C}$

(ii) If A in **normal** then $d\Gamma(A)$ is normal.

(iii) If A is **essentially self-adjoint** then $d\Gamma(A)$ is e.s.a.

(iv) If A is **positive** then $d\Gamma(A)$ is positive.

$d\Gamma$ does *not* preserve boundedness. When A is the identity on $\mathcal{H}$, $d\Gamma(I)$ is the **total number operator**. The vector $\psi_o = (1,0,0,0\ldots)$ (that is, all $\eta^n = 0$ except for $\eta^0 = 1 \in \mathcal{H}^0$) clearly satisfies $d\Gamma(I)\psi_0 = 0$; it defines the **vacuum**.

We define the canonical second quantization of a unitary operator U on $\mathcal{H}$ as that operator on $\mathfrak{F}_S(\mathcal{H})$ which equals $\otimes_{i=1}^n U$ when restricted to $\mathcal{H}_S^n$, $n>0$, and which equals the identity on $\mathcal{H}^0$. It is denoted $\Gamma(U)$ and is unitary. This notation is convenient, for if U_t is a continuous 1-parameter unitary group on $\mathcal{H}$ then by Stone's theorem there exists an e.s.a. operator A on $\mathcal{H}$ such that:

$$U_t = e^{-iAt}. \tag{30}$$

On the other hand, $\Gamma(U_t)$ is also a continuous 1-paramter

[26] Segal and Goodman [1965].

unitary group on $\mathfrak{F}_S(\mathcal{H})$ with generator $d\Gamma(A)$, *i.e.* we can write $\Gamma(U_t) = e^{-id\Gamma(A)t}$.

All of the foregoing applies without modification in the antisymmetric case, except that S_n is replaced everywhere by A_n and $\mathfrak{F}_S(\mathcal{H})$ by $\mathfrak{F}_A(\mathcal{H})$. The following suffers minor modifications in the fermion case, and from now on we treat the two separately.

Symmetric case

We define the creation and annihilation operators as follows; for vectors η in $\mathcal{H}_S^n$ of the form $\psi_1\otimes\psi_2\otimes\ldots\otimes\psi_n$ we define the map $b^-(f)$: $\mathcal{H}^n \longrightarrow \mathcal{H}^{n-1}$ for any $f\in\mathcal{H}$

$$b^-(f)\eta = (f,\psi_1)\psi_2\otimes\ldots\otimes\psi_n. \tag{31}$$

For $f = u_k$ this map is precisely $\exp(-i\Theta_k/\hbar)$, as defined in (1.2.3) Eq.(22). $b^-(f)$ extends to a unique bounded linear map of norm $|f|$ of $\mathcal{H}^n$ into $\mathcal{H}^{n-1}$, where we assume $\mathcal{H}^0$ is mapped onto zero, and in fact preserves symmetry; therefore, $b^-(f)$ is a linear operator on $\mathfrak{F}_S(\mathcal{H})$. Its adjoint $b^+(f) = b^-(f)^*$ has the action $b^+(f)$: $\mathcal{H}^n \longrightarrow \mathcal{H}^{n+1}$ given by

$$b^+(f)(\psi_1\otimes\psi_2\otimes\ldots\otimes\psi_n) = f\otimes\psi_1\otimes\psi_2\otimes\ldots\otimes\psi_n \tag{32}$$

and it does *not* preserve symmetry. The **annihilation operator** $a(f)$ is defined on $\mathfrak{F}_S(\mathcal{H})$ with domain F_0 as:

$$a(f) = b^-(f)N^{1/2} = (N+1)^{1/2}b^-(f) \tag{33}$$

The **creation operator** may be defined as its adjoint restricted to F_0; one finds for $\psi,\phi \in F_0$ that:

$$((N+1)^{1/2}b^-(f)\psi,\phi) = (\psi, Sb^+(f)(N+1)^{1/2}\phi)$$

so that

$$a^*(f) = Sb^+(f)(N+1)^{1/2} = SN^{1/2}b^+(f). \tag{34}$$

Unlike the operators $b^\pm(f)$, the creation and annihilation operators are unbounded. They are closeable and we denote their closure by the same symbols. From their definition they satisfy the commutation relationships:

$$[a(f),a^*(g)] = (f,g) \tag{35}$$

Further, for any unitary operator U on $\mathcal{H}$ it follows that:

$$\Gamma(U)a(f)\Gamma(U)^{-1} = a(Uf),\ \Gamma(U)a^*(f)\Gamma(U)^{-1} = a^*(Uf). \tag{36}$$

The (unbounded) operator given by

$$N(f) = a^*(f)a(f) \tag{37}$$

is e.s.a., preserves symmetry, and vanishes on elements of the form $\psi_1 \otimes \psi_2 \otimes \ldots \otimes \psi_i \otimes \ldots . \otimes \psi_n \in \mathcal{H}^n$ when $(f, \psi_i) = 0$ for all i, and has eigenvalue k on $S\psi_1 \otimes .. \otimes \psi_i \otimes .. \otimes \psi_n \otimes f_1 \otimes \ldots \otimes f_k \in \mathcal{H}_S^{n+k}$. It is called the **number operator for the state** f. It follows from Eq. (36) that:

$$\Gamma(U)N(f)\Gamma(U)^{-1} = N(Uf). \tag{38}$$

There is a fundamental property of the creation and annihilation operators which will be of great importance in the sequel, particularly Sections 3.3 and 3.4. The creation operator is **complex linear** in its argument; the annihilation operator is **complex antilinear**. That is:

$$a(\lambda f) = \bar{\lambda} a(f); \quad a^*(\lambda f) = \lambda a^*(f) \tag{39}$$

for $\lambda \in \mathbb{C}$. These properties follow trivially from their definition; as a consequence the number operator for the state f satisfies:

$$N(\lambda f) = |\lambda|^2 N(f)$$

We define the **Segal field** as the operator:

$$A(f) = (\hbar/2)^{1/2}(a(f) + a^*(f)) \tag{40}$$

It enjoys the following fundamental properties:

(i) It is **real (but not complex) linear** in its argument.

(ii) It obeys the commutation relationship:

$$[A(f), A(g)] = i\hbar \, \mathrm{Im}\,(f,g) \tag{41}$$

($\mathrm{Im}(.,.)$ is the imaginary part).

(iii) It is e.s.a; $A(f)^* = A(f)$.

(iv) The vacuum ψ_0 is in the domain of all finite polynomials in the Segal field, and the vectors $A(f_1)A(f_2)\ldots A(f_n)\psi_0$ are dense in $\mathfrak{F}_S(\mathcal{H})$.

All of these properties flow directly from the definitions of the creation and annihilation operators, as does:

$$\begin{aligned} a(f) &= (2\hbar)^{-1/2}(A(f) + iA(if)) \\ a^*(f) &= (2\hbar)^{-1/2}(A(f) - iA(if)) \end{aligned} \tag{42}$$

One can, however, reverse this methodology; the field A will be of fundamental importance in the abstract approach to QFT, and from it the creation and annihilation operators

will be *defined* by Eq.(42). For the moment we only note that if we define the quantities

$$W(f) = e^{iA(f)/\hbar} \tag{43}$$

then:

$$W(f)W(g) = e^{i\ \mathrm{Im}(f,g)/2\hbar}\ W(f+g) \tag{44}$$

Eq.(44) defines the **Weyl algebra over** $\mathcal{H}$.

We give the example of the foregoing, with which we shall be concerned throughout. Let $\mathcal{H} = L^2(M,d\mu)$ with inner product $(f,g) = \int \overline{f(m)}g(m)d\mu$ and define $\mathcal{H}^n$, $\mathfrak{F}_S(\mathcal{H})$ as above. For $\psi \in \mathfrak{F}_S(\mathcal{H})$, ψ^n is the projection of ψ on $\mathcal{H}^n_S$; ψ^n is of the form $\psi^n(m_1,m_2,\ldots,m_n)$ and is symmetric in all its arguments. The operators $a(f)$, $a^*(f)$ for $f\in\mathcal{H}$ then have the action:

$$\begin{aligned}(a(f)\psi)^n(m_1,m_2,..,m_n) &= \sqrt{n+1}\int \overline{f(m)}\ \psi^{n+1}(m\ ,m_1,\ldots.,m_n)\ d\mu \\ (a^*(f)\psi)^n(m_1,m_2,..,m_n) &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n} f(m_i)\ \psi^{n-1}(m_1,..\hat{m}_i...,m_n)\end{aligned} \tag{45}$$

in which we have written $d\mu$ for the product measure on $\mathcal{H}^n$ also, and $\hat{m}_i$ means that m_i is omitted. As our second example, we consider the same space but in the **occupation number representation**. For this we need a basis $\{u_k\}$ for $\mathcal{H}$, where the index k runs over the positive integers. We then define the orthonormal basis for $\mathfrak{F}_S(\mathcal{H})$ as the vectors:

$\phi_{n_1n_2\ldots n_k\ldots}$ where $\Sigma_k n_k < \infty$ by:

$$\left[\frac{(\Sigma_k n_k)!}{\Pi_k\ n_k!}\right]^{1/2} S_{\Sigma_k n_k}\ \underbrace{u_1\otimes\ldots\otimes u_1}_{n_1}\otimes\underbrace{u_2\otimes\ldots\otimes u_2}_{n_2}\otimes\ldots.\otimes\underbrace{u_k\otimes\ldots.\otimes u_k}_{n_k}\otimes\ldots. \tag{46}$$

An arbitrary vector ψ in $\mathfrak{F}_S(\mathcal{H})$ can then be expanded:

$$\psi = \sum_{n_1\ldots n_k\ldots} C_{n_1\ldots n_k\ldots}\phi_{n_1\ldots n_k\ldots}$$

and the action of the annihilation and creation operators on ψ given as:

$$\begin{aligned}a(u_i)\psi &= \sum_{n_1\ldots n_k\ldots} C_{n_1\ldots n_k\ldots}\, n_i^{1/2}\phi_{n_1\ldots n_k\ldots n_i-1\ldots} \\ a^*(u_i)\psi &= \sum_{n_1\ldots n_k\ldots} C_{n_1\ldots n_k\ldots}\,(n_i+1)^{1/2}\phi_{n_1\ldots n_k\ldots n_i+1\ldots}\end{aligned} \tag{47}$$

We cannot make sense of creation and annihilation *operators* of the form a(k), a(x), etc, because eigenstates of position and momentum do not exist. Nevertheless given distributions δ_m such that $\int\overline{\delta_m(m_1)}f(m_1)d\mu = f(m)$ for $f \in \mathcal{H}$ Eq.(45) yields,

where we use the notatation $a(\delta_m) = a(m)$, $a^*(\delta_m) = a^*(m)$:

$$(a(m)\psi)^n(m_1,m_2,..,m_n) = \sqrt{n+1}\ \psi^{n+1}(m\ ,m_1,....,m_n)$$
$$(a^*(m)\psi)^n(m_1,m_2,..,m_n) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\delta_m\ (m_i)\ \psi^{n-1}(m_1,..\hat{m}_i...,m_n) \quad (48)$$

(where m is a "generalized eigenvalue" - typically of position or momentum). The transformations $a(\mathbf{k})$, $a^*(\mathbf{k})$ nevertheless make sense as quadractic forms on $\mathfrak{F}_S(\mathcal{H})\otimes\mathfrak{F}_S(\mathcal{H})$, and one has formally:

$$[a^*(m_1),a(m_2)] = \delta_{m_1}(m_2). \quad (49)$$

We shall use these expressions in Section 3.3 in the relativistic case, when M is the positive mass hyperboloid and $d\mu$ is the measure d^3p/p_0, where $p_0 = +(\mathbf{p}^2+ m^2c^2)^{1/2}$. We note that the creation and annihilation operators $a^*(f)$, $a(f)$ may be formally recovered from the $a(\mathbf{k})$'s by the integral expressions:

$$a(f) = \int_M a(m)\ \overline{f(m)}\ d\mu$$
$$a^*(f)= \int_M a^*(m)f(m)\ d\mu \quad (50)$$

Antisymmetric case

The foregoing applies to the antisymmetric case with the following modifications. The operators $b^{\pm}(f)$ acting on $\mathfrak{F}_A(\mathcal{H})$ are defined as in Eqs.(31) and (32). The fermion annihilation and creation operators are similarily defined as in Eq.(33), (34), with the antisymmatrizer A in place of S, on the domain (on the domain $F_0 \subseteq \mathfrak{F}_A(\mathcal{H})$) but in place of the commutation relationships Eq.(35) it is a *consequence*[27] of these definitions that:

$$[a(f),a^*(g)]_+ = a(f)a^*(g) + a^*(g)a(f) = (f,g)$$
$$[a(f),a(g)]_+ = [a^*(f),a^*(g)] = 0 \quad (51)$$

These operators also obey Eq.(36) and satisfy the same

[27] This can be seen from their action in the occupation number representation (Eq.(52)). That the creation and annihilation operators defined on the antisymmetric Fock space must obey anticommutator relationships is a not so obvious result that cost Jordan much effort; the converse, that states built up from such operators obeying the ACR's must be antisymmetric, is trivial.

linearity properties as in the boson case. The number operator for the state f is defined as in Eqs.(37) and obeys Eq.(38). The Segal field has now a different rôle in the abstract formulation of the theory and is defined with the omission of the factor in $\hbar$:

$$A(f) = (2)^{1/2}(a(f) + a^*(f)) \tag{52}$$

In place of Eq.(41) we find that:

$$[A(f),A(g)]_+ = \mathrm{Re}\ (f,g) \tag{53}$$

Unlike the boson case, the fermion creation and annihilation operators are bounded. As a consequence the number operator for the state f, as also the Segal field, are bounded self-adjoint operators. The Segal field generates[28] the *Clifford algebra* over the space $\mathcal{H}$, which is the natural abstract object for fermionic fields, in place of the Weyl algebra for bosonic fields.

We consider the same example as before, with the obvious modifications. In place of Eq.(45) we obtain:

$$(a(f)\psi)^n(m_1,m_2,..,m_n) = \sqrt{n+1}\int \overline{f(m)}\ \psi^{n+1}(m\ ,m_1,\ldots,m_n)\ d\mu_o$$
$$(a^*(f)\psi)^n(m_1,m_2,..,m_n) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(-1)^{i-1}f(m_i)\ \psi^{n-1}(m_1,..\hat{m}_i...,m_n) \tag{54}$$

We may also define the bilinear forms as in Eq.(48). Eq.(50) also follows for the anticommutator in place of the commutator.

The occupation number representation is defined as before, except that the antisymmetrizer is used and the factors in $n_i!$ are redundant, since for any given state the occupation number may be 1 at most. For the same reason the square roots in Eq.(47) are redundant, and n_i in the second of these equations must be zero if the corresponding term is non-vanishing. With these modifications, and taking into account the phase change due to antisymmetrization, we

[28] Generate, that is, in the sense that the algebraic operations applied to the A(f)'s, for all $f \in \mathcal{H}$, generate the algebra; not in the sense of Stone's theorem. We shall be dealing with algebras of bounded operators; in the fermion case the fields themselves are in the algebra, in the boson case they are not.

obtain:

$$a(u_i)\psi = \sum_{n_1\ldots n_k\ldots} C_{n_1\ldots n_k\ldots}\,(-1)^{s(i)} n_i \phi_{n_1\ldots n_k\ldots n_i-1\ldots}$$
$$a^*(u_i)\psi = \sum_{n_1\ldots n_k\ldots} C_{n_1\ldots n_k\ldots}\,(-1)^{s(i)}(1-n_i)\,\phi_{n_1\ldots n_k\ldots n_i+1\ldots} \qquad (55)$$

Where $s(i) = \sum_{k=1}^{k=i-1} n_k$ is the number of occupied states up to i.

We conclude with some brief comments on the transformation theory. First observe that for an orthonormal basis $\{u_i\}$ the bounded operator $P_{ij} : f \longrightarrow (u_i, f)u_j$ has canonical second quantization $d\Gamma(P_{ij}) = a^*(u_j)a(u_i)$. Since for an arbitrary linear operator A on $\mathcal{H}$ we may write $Af = \sum_{ij} (u_j, Au_i)P_{ij}f$ for all $f \in \mathcal{H}$, it follows by complex linearity of $d\Gamma$ that:

$$d\Gamma(A) = \sum_{ij}(u_j, Au_i)\Gamma(P_{ij}) = \sum_{ij} (u_j, Au_i)a^*(u_j)a(u_i) \qquad (56)$$

which is Eq.(25). Formulae of the type Eq.(23) are not so easily dealt with; I refer to Cook [1953] for details. They can in fact be placed on a rigorous basis. In this thesis we shall wherever possible avoid use of the point fields; we only mention here, that the vanishing of the equal-time CCR's for the point fields at distinct points in physical space depends critically on the existence of a *local* configuration space inner product (*e.g.* of the form $\int \overline{f(m)}g(m)d\mu$), which vanishes when the supports of f,g have zero intersection. This will be discussed in detail in Part 3, particularly Sections 3.2, 3.3, and 3.4.

Section 1.4. Relativistic Quantum Theory

> I think that this discovery of antimatter was perhaps the biggest jump of all the big jumps in physics in our century.
>
> W. Heisenberg, 1972

1.4.1. Introduction.

The object of this section is to state some clearly defined goals, in the context of the interpretation of RQFT *circa* 1934. This period (1933-1935) marks a watershed in the development of relativistic quantum theory. On the one hand, the fundamental structure of the free field theory, for spin 1/2 and spin 0 systems, had been laid down, and is presented in intoductory texts today much as it was then. The early controversies over the interpretation of NRQM had disappeared, and such difficulties of interpretation of the relativistic theory that had been raised lay quiescent. As far as physicists were concerned, there existed an exciting new realm of phenomenology - opened up by the discovery of cosmic rays, and the development of particle accelerators - and a ready (largely figurative) formalism to hand which appeared to describe the qualitative features of pair creation and annihilation, and even single creation and annihilation processes, based on perturbation theory about the free field theory. For the next two decades the focus of research centered on the perfection of these methods and the development of the renormalization theory.

On the other hand the systematic investigation of the conceptual and mathematical foundations of quantum theory was just beginning. Important and influential treatises by Weyl and von Neumann were beginning to exert a slow but

inexorable effect on the way that the mathematical formalisms of physics should be interpreted. Throughout the thirties and forties the number of people involved in this endeavour, in applications to physics, (what I shall call the *abstract method*) remained small - no more than a handful. But with the likes of Jordan, von Neumann, and Wigner, soon to be joined by Segal and Mackey, much progress was made, so that by the 1950's a durable and solid conceptual framework was laid down.

It would be too much to say that there followed an explosion of interest in their ideas. But by the 1960's a clear research programme became discernable spanning three disciplines: mathematics, physics, and philosophy of science. At the sharp end of each were the fields: C^*-algebras, constructive quantum field theory, and quantum logic, and all three have intensified as thriving fields into the 1970's. In the 1980's the situation is dramatically changed: in the philosophy of physics, the quantum logic approach - *qua* logical realism - has produced little of interest for more than a decade, whilst the proof that $\lambda\phi^4$ theory is trivial in 3+1 dimensions has left the constructive quantum field theorists in disarray. Meanwhile the grand unification programme in physics and string theory have opened up radically new branches of mathematics, particularly in algebraic topology, with an attendent phenomenology which has little to do with scattering phenomena, but concerns rather cosmology on the one hand, and general features of the world - like chirality, the dimensionality of spacetime, the mass spectra - on the other.

This marks a new *rapprochement* between the physics and mathematics communities, for with few exceptions physicists had remained almost totally ignorant of the enormous progress made in the general theory of operator algebras and the representation theory of non-compact Lie groups, theories which had their origins in the first development of abstract methods in physics. The watershed, the growing

divide between physics and mathematics, may be traced to the mid 1930's.

In the complex and chaotic period from 1927-1935, I shall distinguish the following elements (roughly in chronological order):

(i) the invention of relativistic quantum mechanics, and the realization of its disimilarities with the NRQM and its ultimate incoherence except in special (weak external field) interactions.

(ii) early discussions of the Born interpretation for the relativistic theories.

(iii) the covariant quantization of the radiation field.

(iv) the invention of the hole theory.

(v) new phenomenology; the positron, cosmic rays, meson physics.

(vi) the devlopment of massive RQFT, and its assimilation of the hole theory.

(vii) the emergence of renormalization theory.

(viii) the emergence of the abstract approach.

Of these I shall only be concerned with (i), (ii), (iv), (vi) and (viii). Of these (i), (iv) and (vi) have a close logical connection and I shall discuss these first. The historical background will be kept to a minimum. I shall then consider (ii) and (viii) in rather more detail, and conclude with a statement of objectives.

A note on relativistic notation. In this section only, we use greek symbols μ,ν,σ to range over $0,1,2,3$ with Einstein summation convention; the metric g has negative signature. x^μ is a contravariant vector with components $(ct, x_1, x_2, x_3) = (x_0, \mathbf{x})$, $\partial/\partial x^\mu = \partial_\mu$ is a covariant vector with components $(\partial/c\partial t, \partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$; the covariant 4-momentum is $p_\mu = (E/c, -\mathbf{p})$ and the covariant 4-potential $A_\mu = (A_0, \mathbf{A})$. p.x will mean $x^\mu p_\mu$, and $\gamma.\nabla$ will mean $\Sigma_{i=1}^{3} \gamma^i \partial_i$.

1.4.2. Relativistic quantum mechanics.

RQM was invented even prior to the non-relativistic wave mechanics; when Schrödinger sought a massive wave equation, newly emerged from his study of quantum statistics (1.1.4), he naturally looked for a wave equation familiar from classical physics, that is *second order* in the time derivative. It was his inspiration to consider the Hamilton-Jacobi equation and replace the momentum and energy by derivatives in position and time respectively:

$$\left(\partial S/\partial x_\mu - \frac{e}{c}\phi^\mu\right)\left(\partial S/\partial x^\mu - \frac{e}{c}\phi_\mu\right) - m^2c^2 = 0$$

$$\partial S/\partial x^\mu \longleftrightarrow p_\mu$$

$$\left(p^\mu - \frac{e}{c}A^\mu\right)\left(p_\mu - \frac{e}{c}A_\mu\right) - m^2c^2 = 0$$

$$S \longleftrightarrow -i\hbar \ln \psi \qquad p_\mu \longleftrightarrow i\hbar\partial/\partial x^\mu$$

$$\left[\left(i\hbar\partial^\mu - eA^\mu\right)\left(i\hbar\partial_\mu - eA_\mu\right) - m^2c^2\right]\psi = 0$$

Where I have attempted, somewhat figuratively, to indicate the principal lines of reasoning involved. Schrödinger in fact had considerable difficulty in reconciling himself to the non-relativistic equation, first order in time, and intrinsically *complex*[1] (that is, admitting no real solutions), quite apart from the violation it does to the de Broglie theory, in which the phase is considered a geometric quantity, and which was an an intrinsically relativistic theory[2].

[1] We shall eventually (Section 3.4) get an idea of why this is so. There is ample evidence that Schrodinger tried hard to interpret the non-relativistic equation in terms of the relativistic one, or at least in terms of a second-order equation - even to the point of considering its square, deriving a second order equation in time, but forth order in the spatial derivatives.

[2] In an extraordinary paper by Mackinnon [1976], it is argued that the phase waves of the non-relativistic theory, but not those of the relativistic theory, satisfy the basic

The Klein-Gordon equation (KG equation), as it was soon to be called, was so natural an object that it was independently discovered many times in 1926, and in the four month period from April 1926 appeared in independent publications by Schrödinger, Klein, Fock, de Donder and van Dungen, Gordon, de Broglie, and Kudar; alternative derivations continued to trickle in for several months (*e.g.* Iwanenko and Landau [1926], Guth [1927]).

The most ambitious interpretation of the new equation is due to Klein [1926a] in the context of the 5-dimensional unification of Maxwell theory and General Relativity due to Kaluza [1921]. Having derived the Hamilton-Jacobi equation for geodesic motion on the 4+1 dimensional manifold (the fifth coordinate being distinguished), he employed the substitutions $p_k \longleftrightarrow i\hbar\, \partial/\partial x^k$ ($k = 0, 1,2,3,4$) and assumed ψ separable in the fifth coordinate. With solutions of the form $\psi = \phi(\mathbf{x},t)e^{\pm ip_4 x}$ the (constant) canonically conjugate momentum p_4 appeared in the equation of motion along with the 4-potentials, and was identified as proportional to the *charge*. Corresponding to the two signs of the solution, Klein considered the system described both the proton ($p_4 \propto e$) and the electron ($p_4 \propto -e$):

> The fact that with a single parameter one can obtain two distinct classes of waves, which behave so to speak as the positive and negative electrical particles, can be understood with hindsight: it is actually possible, to formulate the wave equation, so that the dynamical behaviour of both kinds of particles are described as a single system. (Klein [1926a p.902-3]).

In his second note (Klein [1926b]) in which he proposed to quantize the charge through the hypothesis that the fifth dimension is compact, he was more specific:

requirements of the de Broglie theory. The arguments are completely falacious and ignore the fundamental point, that the phase of a solution to the relativistic equation, but not the non-relativistic equation, can be regarded as a geometric invariant. This has its origin in the fact that the non-relativistic equation defines a projective representation of the Galilean group (see Section 2.4).

> If the five-dimensional space is assumed to be closed in the direction of x^4 with a period l, and if we apply the formalism of quantum mechanics to our geodesics, we shall expect p_4 to be governed by the following rule:
>
> $p_4 = N\ h/l$
>
> N being now a quantum number, which may be positive or negative according to the sense of motion in the direction of the fifth coordinate. (Klein [1926b p.516]).

We see that at the very inception of the wave mechanics the fundamental idea that a *single* wave equation describe particles of *both* signs of the charge. The theory was, moreover, enormously influential; de Broglie, Pauli, Ehrenfest, Wiener, and Einstein, amongst others, were soon to make detailed contributions. And this idea arose not in the context of the Dirac theory, nor did it have anything to do with negative energy states.

In more conventional treatments, it was scarcely even acknowledged that the wave equation possessed negative frequency solutions. In the detailed development of a semi-classical theory of QED in the hands of Gordon, (Gordon [1926]) the problem was not once mentioned. It seems that only Dirac showed more than a perfunctory interest in the basic structure of quantum mechanics[3].

The difficulties of interpretation arise not only because the KG equation possesses negative frequency solutions, which would seem to correspond to negative energy states, but also because the natural sesquilinear form[4]:

$$\langle \phi_1, \phi_2 \rangle = i\hbar \int \overline{\phi_1(x)} \overleftrightarrow{\partial_t} \phi_2(x)\ d^3x \qquad (1)$$

is not positive definite. Yet the integrand is the only time-component of a divergence-free 4-vector, *i.e.* the probability-current vector:

[3] 1927 Solvay conference: Bohr: "What are you working on?" Dirac: I'm trying to get a relativistic theory of the electron." Bohr: "But Klein has already solved that problem." (Wiener [1977 p.109]. Bohr's aversion to purely formal considerations is well-known; it is not so often considered, that he was also inattentive to the logical structure of the quantum theory.

[4] Our notation is conventional; $f \overleftrightarrow{\partial} g = f\partial g - (\partial f)g$.

$$\rho^{\mu}(x) = ich\,\overline{\phi(x)}\overleftrightarrow{\partial_{\mu}}\phi(x) \qquad (2)$$

which satisfies $\partial_{\mu}\rho^{\mu} = 0$ by virtue of the KG equation for vanishing external field[5]. Further, whilst Schrödinger's initial prejudice against complex-valued wave functions would naturally motivate the assumption of real-valued ϕ, this condition is not possible when an external field is applied; the Cauchy data for a *complex* ϕ then requires the specification of the (complex) values of ϕ, together with the (complex) values of its first derivative. In NRQM the specification of the (complex) values of ψ alone is sufficient.

These difficulties were summarized by Dirac in his third major contribution to physics, together with the objection that it did not seem possible to develop a transformation theory on the basis of the KG equation. One has, in effect, only a conserved 4-current (which Gordon actually interpreted as a charge-current vector): this, Dirac ceded, may be satisfactory so far as the emission and absorption of radiation are concerned, but

> ...is not so general as the interpretation of NRQM, which has been developed sufficiently to enable one to answer the question: what is the probability of any dynamical variable at any specified time having a value between any specified limits, when the system is represented by a given wave function ψ? The Gordon - Klein interpretation can answer such questions if they refer to the position of the electron (by the use of the ρ_4) but not if they refer to its momentum, or angular momentum or any other dynamical variable. We should expect the interpretation of the relativity theory to be just as general as the non-relativity theory.
> The general interpretation of NRQM is based on the transformation theory, and is made possible by the wave

[5] The fact that one must make this restriction already indicates a profound difference between the relativistic and non-relativistic case. In NRQM the probability flux $\rho = (\bar{\psi}\psi,$ $(\hbar/2im)(\bar{\psi}\nabla\psi - (\nabla\bar{\psi})\psi))$ is divergence-free even in the presence of an external potential. The sesquilinear form $($ of Eq. (2) is not invariant under the time evolution in the presence of an external potential. One can define a conserved probability flux in the general case, which depends on the external potentials; defining the sesquilinear form in this way leads to a time-dependent form, taylored to the interaction.

equation being of the form
$(H-W)\psi = 0$
i.e. being linear in W or $\partial/\partial t$, so that the wave function at any time determines the wave function at any later time. The wave equation of the relativity theory must also be linear in W if the general interpretation is possible. (Dirac [1928a p.611-2])[6].

Dirac's critique is misleading ; the Gordan-Klein interpretation provides an adequate probabilistic account of *momentum* particle distributions, not positions. No-one at this time had considered the self-adjointness of the operators $x_i\phi(x)$ $(i=1,2,3)$ with respect to the sesquilinear form Eq.(1), nor had they considered defining momentum-space positive frequency solutions through the integral representation[7]:

$$\begin{aligned}\phi(x) &= (2)^{1/2}(2\pi\hbar)^{-3/2}\int f(p)e^{-ip.x/\hbar}\delta(p^2-m^2c^2)\theta(p_0)d^4p \\ &= (2\pi\hbar)^{-3/2}\int_{\mathbb{P}^3} f(\mathbf{p})e^{-ip.x/\hbar}d^3p/\sqrt{2}p_0\end{aligned} \qquad (3)$$

where $f(p)$ is a suitably well-behaved function on $\mathbb{P}^4$, and $f(\mathbf{p})$ is the function on $\mathbb{P}^3$ defined by $f(\mathbf{p}) = f((\mathbf{p}^2+m^2c^2)^{1/2},\mathbf{p})$; in expressions of this form (*i.e.* the second of Eq.(3)) p_0 is always understood as the function $(\mathbf{p}^2+m^2c^2)^{1/2}$ on $\mathbb{P}^3$. $\theta(y)$, for $y \in \mathbb{R}$, is the function $\theta(y) = 1$ for $y > 0$, and 0 otherwise. The sesquilinear form for $\phi(x)$ defined in this way becomes:

$$\langle\phi_1,\phi_2\rangle = \int_{\mathbb{P}^3}\overline{f_1(\mathbf{p})}f_2(\mathbf{p})d^3p/p_0 \qquad (4)$$

that is, when Eq.(1) is restricted to positive frequency solutions. We now have a positive definite sesquilinear form, and we can make the space of positive frequency solutions into a Hilbert space in the usual way. The operators given by:

$(k_if)(\mathbf{p}) = p_if(\mathbf{p})$, $i = 1,2,3$

are clearly e.s.a. with respect to $\langle.,.\rangle$. They define the

[6] Dirac continued with a discussion of the negative frequency solutions; we shall come on to this shortly. Later he was to remark: "The transformation theory had become my darling. I was not interested in considering any theory which would not fit in with my darling."(Pais [1986 p.288]).

[7] I shall call this transform the covariant Fourier transform, the usual (but not terribly perspicuous) terminology.

one-particle **momentum operators**.

The momentum space representation for the scalar RQM was not developed in any explicit sense at this time, and almost all the attempts to develop a RQM were soon to be centered on the Dirac equation instead. What is remarkable is that a full 20 years were to pass before physicists again considered the 1-particle scalar theory. I shall consider the subsequent development of the theory in Section 3.2, but here I wish to stress two things; first, quite obviously, the foregoing does not provide a RQM in the same sense as NRQM. There is no transformation theory, no position operators, no canonical commutation relationships - and this remained true throughout the 30's and 40's. But second, what I call the abstract method, was already at this time providing a deeper and more general idea of what the NRQM is[8], and considered in *this* sense, a clearly defined RQM was developed in 1939 by Wigner. Prior to this time, and even after, as far as the physics community was concerned[9], it is difficult to attach a precise meaning to the notion of a particle interpretation of RQFT, or to compare the logical structure of NRQM and RQFT.

To return to Dirac's great contribution of 1928[10], it is

[8] I refer to Hermann Weyl's profound study of 1927 "Quantenmechanik und Gruppentheorie"; see (2.4.5).

[9] Wigner's paper was widely known, although I suggest scarcely read or comprehended by the physics community. The summary view, that this paper obtained all possible relativistic wave equations, misses the fundamental insight, basic to this study, as to what a relativistic wave equation actually is. Dirac again was one of the few physicists who embraced, and partly motivated, this insight; see Section 3.1

[10] I have not considered the Pauli theory of spin and the introduction of spinors into NRQM. I refer to Serwer [1977] very throrough account of these developments, and to Kragh [1979] for some wider background on the attempts to find a relativistic version of the spinor theory. I do not myself consider that these played any role in the genesis of the theory, however. Dirac later remarked: "...it was found that (the Dirac equation) gave the particle a spin of half a quantum...that was really an unexpected bonus for me,

clear why he sought a first order relativistically covariant wave equation. Considering the free KG equation:

$(-p_0^2 + \mathbf{p}^2 + m^2c^2)\psi = 0$

(the p's were understood as the operators $i\hbar\partial_\mu$, not as the k's defined above), he demanded an equation of the same order in the p_i's and p_0, first order in p_0. He wrote down the expression:

$(p_0+\alpha^1 p_1+\alpha^2 p_2+\alpha^3 p_3 + mc\beta)\psi = 0$

with the α's ,β and p's all operators, the α's and β's independent of each other and of the p's, and obeying algebraic relationships such that ψ automatically satisfies the KG equation.

A remarkably simple, but effective heuristic. One requires:

$(p_0+\alpha^1 p_1+\alpha^2 p_2+\alpha^3 p_3 + mc\beta)(-p_0 + \alpha^1 p_1 + \alpha^2 p_2 + \alpha^3 p_3 + mc\beta) = -p_0^2 + \mathbf{p}^2c^2 + m^2c^2$

It is elementary to show that this equation is satisfied if:

$\alpha^i\alpha^j + \alpha^j\alpha^i = 2\delta^{ij}$, $\alpha^i\beta + \beta\alpha^i = 0$, $i,j = 1,2,3.$

$\beta^2 = 1$ (5)

or defining $\gamma^0 = \beta$, $\gamma^i = \beta\alpha^i$ simply if:

$[\gamma^\mu, \gamma^\nu]_+ = 2g^{\mu\nu}$ (6)

Quantities γ satisfying Eq.(6) define a **Clifford algebra**. Their properties were soon to become an essential part of the repertoire of quantum physics. The **Dirac equation** now takes the form:

$(-i\hbar\gamma^\mu\partial_\mu + mc)\psi(x) = 0$ (7)

and for an external potential A_μ, introduced as usual via the substitution $p_\mu \longleftrightarrow i\hbar\,\partial_\mu - \frac{e}{c} A_\mu$:

$$\left[\gamma^\mu\left(i\hbar\,\partial_\mu - \frac{e}{c} A_\mu(x)\right) - mc\right]\psi(x) = 0 \qquad (8)$$

Assuming that the α's and β are finite dimensional matrices, one can further deduce that the α's and β must be Hermitian, hence γ^0 Hermitian and γ^i anti-Hermitian (*i.e.* $\gamma^{i*}=-\gamma^i$). The α^i's therefore have exactly the same properties as the Pauli spin matrices σ^i. There is, however, no independent matrix to represent β: not in 2-dimensions, nor indeed in 3, but in a minimum of 4-dimensions. Considering the p's as scalars on this 4-dimensional representation space of the

completely unexpected." (Quoted in Pais [1986 p.286]).

γ's, the wave function ψ must be a 4-component object. In this way the Dirac *bi-spinor* was invented.

The remarkable properties of the Clifford algebra only begin to surface with Dirac's proof of the relativistic covariance of the new wave equation[11]. The standard proof runs as follows: under a Lorentz transformation Λ: $x \longrightarrow x'(x)$ it follows that $\partial/\partial x^{\mu}$ transforms as:

$\partial/\partial x^{\mu} \longrightarrow \partial/\partial x'^{\mu} = (\partial x^{\nu}/\partial x'^{\mu})\partial/\partial x^{\nu} = \Lambda^{\nu}_{\ \mu}\partial_{\nu}$

with inverse:

$\partial_{\mu} = (\partial x'^{\nu}/\partial x^{\mu})\partial/\partial x'^{\nu} = A_{\mu}^{\ \nu}\partial'_{\nu}$.

Under the same transformation suppose that the γ's are unaltered but that the wave-function $\psi(x)$ transform as:

$$\psi(x) \longrightarrow \psi'(x') = S(\Lambda)\psi(x) \qquad (9)$$

where $S(\Lambda)$ is a matrix representation of the Lorentz group. The Dirac equation in the new coordinate system is:

$(-i\hbar\gamma^{\mu}\partial/\partial x'^{\mu} + mc)\psi'(x') = 0$

which can be written

$(-i\hbar\gamma^{\mu}\Lambda^{\nu}_{\ \mu}\partial/\partial x^{\nu} + mc)S(\Lambda)\psi(x)$

so that if $S^{-1}(\Lambda)\gamma^{\mu}S(\Lambda) = \gamma^{\sigma}\Lambda_{\sigma}^{\ \mu}$, then on multiplying on the left by the matrix S^{-1} we obtain

$(-i\hbar\gamma^{\sigma}\Lambda_{\sigma}^{\ \mu}\Lambda^{\nu}_{\ \mu}\partial/\partial x^{\nu} + mc)\psi(x) = (-i\hbar\gamma^{\nu}\partial/\partial x^{\nu} + mc)\psi(x)$.

The Dirac equation therefore transforms covariantly. For later reference, we note the transformation properties of the covariant γ matrices:

$$S^{-1}(\Lambda)\gamma_{\mu}S(\Lambda) = \gamma_{\sigma}\Lambda^{\sigma}_{\ \mu}. \qquad (10)$$

It is easy to see that the matrix representation $S(\Lambda)$ satisfies the property:

$$S(\Lambda)^{*}\gamma^{0}S(\Lambda) = \gamma^{0}. \qquad (11)$$

It is, therefore, non-unitary[12].

[11] Dirac's proof was incomplete. He returned to the question of relativistic covariance in his second paper on the electron theory, in [1928b], where he established the existence of a divergence-free 4-vector (see below).

[12] Some idea of just how remarkable the Clifford algebra actually is can be seen as follows. Consider a real n-dimensional vector space V equipped with a symmetric bilinear from B. The Clifford algebra over V may be defined (heuristically) as the associative algebra generated by the identity and monomials $[u]$, $[v]$, $[w]$, ... where u,v,w ..$\in$ V, such that $[u][v] + [v][u] = 2B(u,v)$. When n is even this algebra is simple and for each n there exists a unique

The divergence-free 4-vector is[13]:

$$\rho^{\mu}(x) = c\ \overline{\psi(x)}^{t}\gamma^{0}\gamma^{\mu}\psi(x) \qquad (12)$$

so that the space integral of its time-component is conserved. The term $\bar{\psi}^{t}\gamma^{0}$ appears frequently in the Dirac theory and we shall denote it $\tilde{\psi}$; it is called the **covariant adjoint**. This notation will be used presently.

From Eq.(12) it follows that the sesquilinear form:

$$\langle\psi,\psi'\rangle = \int \overline{\psi(x)}\psi'(x)d^{3}x \qquad (13)$$

(a summation over spinor components is implict), will be invariant under the time evolution. It is, moreover, positive definite, in contrast to the scalar case, and may be considered an inner product on the solution space at some fixed time. The fact that the equation is first order in time means that the wave function, at some specified time, is sufficient to determine the subsequent evolution uniquely, again unlike the scalar case. For the same reason one can also write the Dirac equation in Hamiltonian form:

$H\psi = i\hbar\partial\psi/\partial t$

where H is the Hamiltonian (for an external field A_{μ}):

$$H = -i\hbar c\gamma^{0}\gamma^{1}\partial/\partial x^{1} + e\gamma^{0}\gamma^{\mu}A_{\mu} + \gamma^{0}mc^{2}. \qquad (14)$$

For all these reasons, it appears, but for the fact that the wave-function has 4 and not 2 components, that the canonical structure of NRQM has been preserved.

But the consideration of the invariance properties of the

irreducible complex matrix representation ρ of this algebra (of dimensionality $2^{n/2}$); for n=4 the representation space may therefore be identified with the complexification of V. The γ matrices are precisely $\rho([e^{i}])$ when B is Minkowski and where e^{i} is a basis for V such that $B(e^{i},e^{j}) = g^{ij}$). If one defines G as the group of invertible transformations L of V such that B(Lu,Lv) = B(u,v) then there exists a unique invertible linear transformation S(L) on $V^{\mathbb{C}}$ such that $S(L)\rho([u])S(L)^{-1} = \rho([Lu])$. For proofs and a brief but beautiful summary of the finite dimensional representation theory of Clifford algebras see Varadarajan [1970 X11.4].

[13] This vector is divergence-free even in the presence of external potentials, unlike the corresponding vector in the scalar case. As a result, the inner product is the same whatever the potential.

inner product under Loretnz transformations reveals that something quite new has been introduced. From Eq.(9) we see that the complex conjugate solutions transform as:

$$\bar{\psi}(x) \longrightarrow \overline{\psi'(x')} = \overline{\psi(x)}S^{*}(\Lambda)$$

so one might think that:

$$\langle\psi,\psi\rangle \longrightarrow \int \overline{\psi(x)}S^{*}(\Lambda)S(\Lambda)\psi(x)d^{3}x.$$

If S were unitary one would obtain invariance, exactly as in NRQM. However we have not allowed for the change in the volume element $d^{3}x$, and S is not unitary; the two cancel each other out[14]. The measure is concentrated on a spacelike hypersurface, which can be specified by a unit timelike vector; under Lorentz transformations, therefore, it transforms as a contravariant vector $d\sigma^{\mu}(x)$. The covariant form of Eq.(13) is:

$$\langle\psi,\psi'\rangle = \int \overline{\psi(x)}\gamma_{0}\gamma_{\mu}\psi'(x)d\sigma^{\mu}(x)$$

And the correct transformation law is:

$$\begin{aligned}\langle\psi,\psi'\rangle &\longrightarrow \int \overline{\psi(x)}S^{*}(\Lambda)\gamma_{0}\gamma_{\mu}S(\Lambda)\psi'(x)\Lambda_{\lambda}{}^{\mu}d\sigma^{\lambda}(x)\\ &= \int \overline{\psi(x)}S^{*}(\Lambda)\gamma_{0}S(\Lambda)S^{-1}(\Lambda)\gamma_{\mu}S(\Lambda)\psi'(x)\Lambda_{\lambda}{}^{\mu}d\sigma^{\lambda}(x) \qquad (15)\\ &= \int \overline{\psi(x)}\gamma_{0}\gamma_{\sigma}\Lambda^{\sigma}{}_{\mu}\psi'(x)\Lambda_{\lambda}{}^{\mu}d\sigma^{\lambda}(x) = \langle\psi,\psi'\rangle\end{aligned}$$

by appeal to Eqs.(10),(11). The inner product itself transforms in such a way as to compensate for the non-unitarity of the representation S[15].

Such mathematical niceties appeared of little concern at the time; the overwhelming priority was to extract empirical confirmation for the theory. In this the theory was at first remarkably successful. Not only did Dirac establish that in a central potential the total angular momentum is a constant of the motion and includes an intrinsic spin s = 1/2 (thereby showing that the theory described the electron), but he also derived the correct magnetic g-factor for the electron, a first approximation to the relativistic

[14] Similar considerations apply to the scalar sesquilinear form. The covariant derivative ∂_{μ} there plays the role of the γ_{μ}.

[15] I shall elaborate on the general structure of this kind of representation in Section 3.3.

Sommerfeld fine-structure formula, the Thomson scattering formula as leading term in scattering theory, and in a suitable approximation obtained the non-relativistic Pauli theory[16]. Both of these latter results were improved upon by Darwin ([1928]), who in particular was able to obtain the exact Sommerfeld formula from the Dirac theory. Shortly after Klein and Nishina [1929] obtained second-order corrections to the scattering theory (Compton scattering), in semi-classical radiation theory.

Only one cloud loomed on the horizon. What of the negative energy states? These too appeared in the Dirac theory, and the cloud loomed more and more darkly.

1.4.3. Negative energy states and the Dirac hole theory.

In one respect Dirac followed Klein in his interpretation of the negative frequency states. In his first paper on the electron theory, he stated two objections, the first discussed above. He continued:

> The second difficulty in Gordon's interpretation arises from the fact that if one takes the conjugate imaginary of (the KG equation) one gets:
> $$\left[(-i\hbar\partial_0 - \frac{e}{c}A_0)^2 + (i\hbar\nabla + \frac{e}{c}\mathbf{A})^2 + m^2c^2\right]\bar{\psi} = 0$$
> which is the same as one would get if one put -e for e. The wave equation thus refers equally to an electron with charge e as to one with charge -e. If one considers for definiteness the limiting case of large quantum numbers one would find that some of the solutions of the wave equation are wave packets moving in the way a particle of charge -e would move on the classical theory, while others are wave packets moving in the way a particle of charge e would move classically. For this second class of solutions p_0 has a negative value. One gets over this difficulty on the classical theory by arbitrarily excluding those solutions that have a negative p_0. One cannot do this on the quantum theory, since in general a perturbation will cause transitions from states with p_0

[16] The ambiguities in defining the non-relativistic limit are of some relevance to the general question of interpretation of the relativistic theory and to the Born interpretation in particular. Because a considerable digression would be involved and because the reasoning is largely formal I shall not pursue it here.

> positive to states with p_0 negative. Such a transition would appear experimentally as the electron suddenly changing its charge from -e to e, a phenomenon which has not been observed. The true relativity wave equation should thus be such that its solutions split up into two non-combining sets, referring respectively to the charge -e and the charge e. (Dirac [1928a p.612].

What was a virtue in 5-dimensions becomes a liability in 4; Klein considered the positive and negative frequency solutions defined with respect to evolution in the fifth coordinate as associated with the electron and proton. Since the motion was geodesic (and in this sense consituted a geometrization of electrodynamics exactly as did GR of the gravitational force) no perturbations need be considered. Further, because the energy is the generator with respect to the time, not the fifth coordinate, the two signs have nothing to do with the energy. Dirac takes over this interpretation in the 4-dimensional theory[17] but now because, on the basis of NRQM, one should be able to apply arbitrary perturbations, and further because the generator is with respect to time, one obtains the unphysical consequences as above[18].

Incidentally, one sees here that Dirac anticipates the theory of 1-particle charge conjugation; Kramers's paper of [1937] no more than elaborated the same observation, but for the Dirac theory.

[17] There is a confusion in the way that Dirac applies this interpretation, which I shall discuss presently. There is also the minor point, that he does not associate the positive charge solutions with the proton at this stage, but (correctly) with a positive charge electron.

[18] Apart from the confusion referred to above, Dirac was essentially correct in his interpretation of the KG equation, but (in the view taken here) mistaken in taking over the canonical formalism of NRQM. In brief, one must deepen the canonical theory to embrace a correspondence between the action of multiplication by complex numbers and a real symplectic transformation on the classical phase space; the negative energy difficulty disappears. At the same time it is no longer possible to introduce arbitrary perturbations, and the quantization is inextricably bound up with the dynamics. This is the one-particle version of the theory developed in Section 3.5.

Dirac ackowledged that the difficulty with negative energy solutions persisted in the new theory. In fact just this difficulty was supposed to explain the appearance of 4-component bi-spinors: the Dirac equation

> ...has four times as many solutions as the non-relativity wave equation, and twice as many as the previous relativity wave equation (the KG equation). Since half the solutions must be rejected as referring to the charge +e on the electron, the correct number will be left to account for duplexity (*i.e.* spin) phenomena. (Dirac [1928a p.618]).

This heuristic has survived to the present day. It is, however, incorrect[19]. We see also the statement that half the solutions "must be rejected". In quantum theory, alas, hand-waving is not nearly so effective as in classical theory.

It should be emphasised that the complex conjugate wave equation does not appear in any natural way in the RQM, no more than does the complex conjugate Schrödinger equation in NRQM (which would also define negative frequency states). The difference is that the KG or Dirac equations admit *both* classes of solutions. Taking the negative frequency solutions, they may be interpreted as negative energy states of charge -e, or as the complex conjugates of positive energy states of charge +e. Dirac adopts a curious line in pointing out that the complex conjugate of *positive* frequency states for charge -e are negative frequency states of charge +e - for this is no problem, and the same is true in NRQM. On one reading it is just false that "in the limiting case of large quantum numbers" one will obtain states of charge +e *and* negative energy - on the contrary one will obtain states of -e and negative energy. On another reading such states will indeed *behave* as states of +e, but with (apparent) *positive energy* (this is only true as regards their behaviour in an external field). We have this argument from Dirac himself the following year:

Let us examine the wave functions representing states of

[19] As I argue towards the end of (3.3.6).

> negative energy a little more closely. If we superpose a number of these wave functions in such a way as to get a wave packet, the motion of this packet will be along a classical trajectory ...such a trajectory, it is easily seen, is a possible trajectory for an ordinary electron (with positive energy) moving in the original electromagnetic field. *Thus an electron with negative energy moves in an external field as though it carries a positive charge.* (Dirac [1929 p.361]; emphasis Dirac).

What had driven Dirac to consider again the negative frequency solutions may have been Klein's proof, that even in a time-independent external potential one cannot satisfy the boundary conditions that are naturally imposed at the boundary of a potential barier of sufficient height, using only positive frequency solutions of the Dirac equation (Klein [1929], the so-called "Klein paradox"). More likely the difficulties in the time-dependent case were already sufficient motivation. Further, in the Klein-Nishina theory of Compton scattering based on the new theory, the formula reduce to the Thomson limit *only* if one sums over intermediate states of *both* signs in the frequency, as observed by Waller[20]. The conclusion is unavoidable: if the initial state is initially a positive-frequency solution, it will develop a negative-frequency part under a time-dependent perturbation. But how to interpret the negative-frequency states in a physically satisfactory way?

We get some idea of how serious this difficulty appeared from a remark of Heisenberg: "the saddest chapter of modern physics is and remains the Dirac theory" (Heisenberg [1928]), adding "it has made Jordan melancholic". This followed a visit of Dirac to Leipzig, where he had discussed the difficulty with Heisenberg. But a solution was to hand, and it stemmed from further reflections on the classical relationships between the charge and the sign of the energy:

[20] Waller reworked the Klein-Nishina theory on the basis of Dirac's QED, using the relativistic Hamiltonian, and noted this fact explicitly (Waller [1930]). Dirac knew of this work, as noted by Pais [1986 p.350] in his comprehensive history of these developments.

> (The result above) has led people to suspect a connection between the negative energy electron and the proton or hydrogen nucleus. One cannot, however, simply assert that a negative-energy electron *is* a proton, as that would lead to paradoxes:
> (i) A transition of an electron from a state of positive to one of negative energy would be interpreted as a transition of an electron into a proton, which would violate the law of conservation of electric charge.
> (ii) Although a negative-energy electron moves in an external field as though it has a positive charge, yet...the field it produces must correspond to its having a negative charge, *e.g.* the negative-energy electron will repel an ordinary positive-energy electron although it is itself attracted by the positive-energy electron.
> (iii) A negative-energy electron will have less energy the faster it moves and will have to absorb energy in order to be brought to rest. No particles of this nature have ever been observed. (Dirac [1929 p.362]).[21]

Dirac's solution was remarkably daring and simple, yet hardly elegant: the negative energy solutions were to be retained, but all - or almost all - the available states were to be considered already occupied. Except in the rare event in which a state is unoccupied, the exclusion principle will prohibit transitions to a negative energy state. Where such a state is unoccupied, these transitions are to be permitted. The vacuum is to replaced by a "sea" of negative energy electrons, of infinite charge, mass, and energy density, which by virtue of isotropy and homogeneity was to be considered empirically unobservable. This is not the first modification to the vacuum that Dirac had introduced, from purely mathematical considerations[22]; but unlike his earlier vacuum, this one led to qualitatively new physics[23].

[21] This critique should be carefully considered in the light of Section 3.5., where we establish that conventional QED, considered as the canonical second quantization of the 1-particle theory, indeed just identifies the positron with the negative frequency solutions. None of Dirac's objections find any purchase, however.

[22] The first being the "zero-frequency sea" of photons, (1.2.4)

[23] New, but not revolutionary. The idea of particle creation and annihilation had been considered by many physicists, amongst them Jordan [1927a], as noted in (1.3.1); see Bromberg [1976] for a review of Dirac's precursors.

As Dirac remarked, this idea - in the context of the atom - is not unfamiliar: it is used in the theory of internal conversion, where an inner electron is expelled from an atom on absorption of α-particles, and subsequent electron transitions to this unoccupied state lead to X-ray emission.

It was remarkable to extend this idea to the physical vacuum; and immediately one is led to a new and rich phenomenology. A transition to an unoccupied negative energy state (hole) will lead to the disappearance of both (positive frequency) electron and hole. The hole itself will behave as the absence of negative charge and negative energy - that is as a positive charge of positive energy. This process will therefore correspond to pair annihilation. Conversely the transition of a negative-frequency electron to a positive frequency state will correspond to the appearance of both the latter and a hole - that is, to pair creation. These positive charged positive frequency particles - the holes - he identified with the proton, the only known positive charge particle. The mass difference was supposed to arise as an effect of many-particle interactions.

The subsequent reinterpretation of the holes - as it became increasingly clear that the mass difference could not be explained in this way - to describe the behaviour of a new kind of particle (the positron) was made by Dirac in [1931][24]. Antimatter was thus predicted on the basis of the hole theory, and its empirical confirmation soon followed[25].

[24] The relevant background is well-documented; see, e.g. Bromberg [1976], Pais [1986]. Hanson [1963] is so confused on the basic facts that its integrity is seriously compromised.

[25] Anderson [1932]. It is not quite a textbook case of experiment guided by theory, however. Anderson later remarked: "Yes, I knew about the Dirac theory...but I was not familiar in detail with Dirac's work. I was too busy operating this piece of equipment to have much time to read his papers....(their) highly esoteric character was apparently not in tune with most of the scientific thinking

I shall only list the subsequent developments[26]:

March 1930: cross-section $e^+ + e^- \longrightarrow 2\gamma$ calculated (Dirac, Oppenheimer, Tamm). The Klein-Nishina formula rederived (Dirac).
May 1931: positron predicted (Dirac). Process $\gamma + \gamma \longrightarrow e^+ + e^-$ suggested to detect positrons.
September 1932. The positron discovered (Anderson).
June 1933: greater efficiency of "internal conversion" suggested. (Oppenheimer, Plesset).
October 1933: first-order calculation of vacuum polarization with attendant charge renormalization (Dirac); non-linear electromagnetism: process $\gamma + \gamma \longrightarrow e^+ + e^- \longrightarrow \gamma + \gamma$ noted (Halpern).
October 1934: pair creation cross-section calculated (Breit and Wheeler).
October 1935: electron-positron scattering cross-section evaluated, including the process $e^+ + e^- \longrightarrow \gamma \longrightarrow e^+ + e^-$ (Bhabba).

The fundamental fact is that by redefining the ground state of the electron theory, the standard ideas of one-particle quantum theory immediately lead to a phenomenology typical of a many-particle theory. In the hole theory pair creation and annihilation became one-particle processes; it is not only that they have this interpretation, but that the one-particle quantum theory could be applied to them. Immediately one could calculate their cross-sections, and quite generally apply the perturbation theory to deduce the existence of more complex processes that proceed by virtual states, such as Bhabba scattering $e^+ + e^- \longrightarrow \gamma \longrightarrow e^+ + e^-$ and Halpern scattering (non-linear electromagnetic effect) $\gamma + \gamma$

of the day...the discovery of the positron was wholly acidental" (quoted in Pais [1986 p.352]).

[26] The earlier results were easily taken over to the electron -positron theory, rather than the electron - proton theory, and I have described them in this way.

$\longrightarrow$ $e^{+}+e^{-}\longrightarrow$ $\gamma+\gamma$. It is often said that the Dirac hole theory transforms the one-particle theory into a many-particle theory: it must be born in mind that it also works the other way, that is, by thinking of the positron as a hole *prima facia* many-particle processes such as pair creation and annihilation can be treated using the formalism of a one-particle theory, that is, as a single particle making transitions from positive to negative energy states (pair annihilation) or from negative to positive energy states (pair creation). In these processes, the negative energy sea plays a purely passive rôle, in restricting the number of negative energy states available for such transitions.

In other processes, however, the sea plays a more active rôle; in particular, the response of the sea to an external field should be just like a dialectric; the negative-energy electrons will be polarized and an induced polarization field will be set up. This is the vacuum polarization, first investigated by Dirac in 1934, and it was here for the first time that the full intricacies of the hole theory were encountered. To deal with them he used a variant of the Hartree self-consistent field method, together with the density matrix formalism. This step leads naturally to the calculation of the effective charge that will produce the "nett" field, that is the external field together with the polarization field of the vacuum; in other words, one is led to the idea of *charge renormalization* (Dirac [1934]).

But even as Dirac was developing new tools, within the framework of quantum statistical mechanics, to handle the intrinsic many-particle character of the new vacuum, others were looking to the techniques of QFT. I have said nothing about the development of RQFT in the intervening period: its evolution, through the years 1928-1934, was an intricate and complex affair, largely because it was developed almost entirely in the context of electromagnetic field theory, with all the attendant problems of gauge invariance. However important, from a purely historical point of view, and also

in the general context of gauge theory, I omit these developments in their entirety. It is sufficient for our purposes to summarize the free Heisenberg-Pauli theory for the Dirac field, and then consider its relationship to the hole theory.

1.4.4. Relativistic Quantum Field Theory[27].

Discussion

In 1929, following Jordan and Pauli's analysis[28] of the free radiation field, it was apparent that a thorough-going relativistic generalization of the Lagrangian field theory of (1.3.3),(1.3.4) could at least formally be defined. Heisenberg and Pauli undertook the task. Most of this paper was concerned with spelling out the details of classical

[27] Most of this subsection is a straightfoward exegesis of the standard formalism of RQFT, scalar and spin half fields. For detailed proofs we refer to Schweber [1961]. For the reader familiar with the theory, it is only necessary to read the sections marked "discussion".

[28] There are three strands to this paper of Jordan and Pauli ([1928]). The first was an attempt to second quantize the free electromagnetic field theory, the second a generalization of the CCR's to unequal times, and the third a preliminary investigation of the representation in which the field itself is "diagonalized".
Of these the first is not directly relevant to our discussion, which is restricted to the massive fields. The second was pursued purely on the basis, of what c-number function (distribution) could stand at the RHS of a commutator of fields which satisfy the wave equations, and which reduces to a delta function at equal times. This analysis was purely formal and may be found in, e.g. Heitler [1954 II.3]. In retrospect one sees the first indication that the free fields are in fact causal i.e. statisfy microcausality; however no such interpretation was proposed, neither at this time nor for many years. The third strand in their account is of great interest; the attempt was, however, abortive and it was many decades before mathematical methods were developed adequate to the task. See (3.5.2) for a rather cursory discussion; this so-called "Schrodinger" representantion is beyond the scope of this thesis.

Lagrangian field theory[29]; the quantization was then implemented by imposing the equal-time CCR's on the canonically conjugate fields. In application to the Dirac field this amounted to defining the classical Lagrangian:

$$L = c \int \Sigma_{ij} \, \psi_i^{\sim}(x) \left(-i\hbar \, \gamma^{\mu}_{ij} \partial_{\mu} + mc \right) \psi_j(x) \, d^3x$$

(where I have explicitly indicated the bi-spinor indices) from which variation with respect to ψ gives the adjoint equation:

$$\psi^{\sim}(-i\hbar\gamma^{\mu}\partial_{\mu} + mc) = 0$$

We also identify the Lagrangian density $\mathcal{L}$:

$$\mathcal{L}(x) = c \, \psi^{\sim}(x) \left(-i\hbar\gamma^{\mu}\partial_{\mu} + mc \right) \psi(x) \qquad (16)$$

from which the canonically conjugate momentum π is obtained:

$$\pi = \partial\mathcal{L}/\partial\dot{\psi} = -i\hbar \, \bar{\psi} \qquad (17)$$

variation with respect to $\bar{\psi}$ just gives us the Dirac equation. The Hamiltonian density is (spinor summation suppressed):

$$\mathcal{H} = \pi \, \dot{\psi} - \mathcal{L} = c\psi^{\sim}(-i\hbar\gamma.\nabla + mc)\psi = \bar{\psi} \, (-i\hbar c\gamma^0\gamma.\nabla + \gamma^0 mc^2)\psi \qquad (18)$$

We begin to discern the outlines of a precise *local correspondence* (in the sense of (1.3.3)) betweeen field and 1-particle theory. Eq.(18) can be written $\mathcal{H} = \bar{\psi} H_0 \psi$, where H_0 is given by Eq.(14) for vanishing external field. Indeed, the correspondence holds true in the interacting case also. The field Hamiltonian $H_F = \int \mathcal{H}(\mathbf{x}) d^3x$ is formally identical to the (configurations space) expectation value of the 1-particle Hamiltonian.

The field is quantized by imposing the equal-time ACR's[30]:

$$[\hat{\pi}_i(\mathbf{x}), \, \hat{\psi}_j(\mathbf{x}')]_+ = -i\hbar \, \delta^3(\mathbf{x}-\mathbf{x}')\delta_{ij} \qquad (19)$$

or in terms of the fields:

$$[\hat{\psi}^*_i(\mathbf{x}), \, \hat{\psi}_j(\mathbf{x}')]_+ = \delta^3(\mathbf{x}-\mathbf{x}')\delta_{ij} \qquad (20)$$

where I have made the bi-spinor indeces explicit. Eq.(20) shows that the $\hat{\psi}$'s, $\hat{\psi}^*$'s have the algebra of annihilation

[29] I assume familiarity with Lagrangian field theory in this thesis; for an introduction see e.g. Goldstein [1974 Ch.11], Schweber [1961 7g], or Bjorken and Drell [1965 Ch.11].

[30] For brevity I shall usually write $\psi(\mathbf{x})$ in equal time ACR's; we work throughout in the Heisenberg picture.

and creation operators, respectively.

The quantum field theory therefore locally corresponds to the 1-particle theory; we have for example that $\hat{H}_F$ and $\langle H_0 \rangle$ are formally identical, except that the configuration space wave-functions that occur in $\langle H_0 \rangle$ are to be replaced by creation and annihilation operators. This is to be expected, because the 1-particle theory is in the canonical form of NRQM. We also note that under the formal prescription:

$d\Gamma(A) = \int \hat{\psi}^*(x) A \hat{\psi}(x) d^3x$

when A is a 1-particle operator in configuration space, that the number operator should be given by

$$N = d\Gamma(\mathbb{I}) = \int \hat{\psi}^*(\mathbf{x}) \hat{\psi}(\mathbf{x}) d^3x \qquad (21)$$

Momentum space expansions

Heisenberg and Pauli reformulated the theory by means of the expansion:

$\hat{\psi}_i(\mathbf{x}) = \Sigma_s a_s u_i^s$

where i is the bi-spinor index and $\{u_i^s\}$ is a 4-component orthonormal basis with the usual properties:

$\int \overline{u_i^s}(\mathbf{x}) u_i^r(\mathbf{x}) = \delta^{rs}$

$\Sigma_s \overline{u_i^s}(\mathbf{x}) u_j^s(\mathbf{x}') = \delta^3(\mathbf{x}-\mathbf{x}')\delta_{ij}$

Where by virtue of the ACR's Eq.(20) one has:

$[a_s^*, a_r]_+ = \delta_{rs}$.

When the basis vectors u^s are later interpreted as energy eigenstates (box normalization), they explicitly indicate that only *positive* energy states are to be considered in the expansion.

We shall use the momentum space expansion:

$$\hat{\psi}(x) = \sqrt{2}(2\pi\hbar)^{-3/2} \int \Sigma_r u(p,r) \hat{a}(p,r) e^{-ip.x/\hbar} \delta(p^2 - m^2c^2) d^4p \qquad (22)$$

where $u(p,r)$ is (for each p) a set of independent bi-spinors. Since $\hat{\psi}$ satisfies the Dirac equation it follows that:

$$(\gamma^\mu p_\mu - mc)u(p,r) = 0 \qquad (23)$$

for each r. Because of the (anti)hermitian properties of the γ 's, Eq.(23), for each p, is a set of 4 real constraints on the 8 real components of each u; therefore only 2 complex

components of each u are independent, and we conclude that r takes values in the index set {1,2}. We may distinguish the positive and negative energy solutions by writing for a positive frequency solution:

$w_r(\mathbf{p}) = u((\mathbf{p}^2+m^2c^2)^{1/2},\mathbf{p},r)$

and for a negative frequency solution:

$w_{r+2}(\mathbf{p}) = u(-(\mathbf{p}^2+m^2c^2)^{1/2},\mathbf{p},r)$

with similar notation for the expansion coefficients b(p,r). In this notation, we perform the p_0 integration in Eq.(22) to obtain:

$$\psi(x) = (2\pi\hbar)^{-3/2}\int_{\mathbb{P}^3}\sum_{r=1,2}\left[w_r(\mathbf{p})\, a(\mathbf{p},r)e^{-ip.x/\hbar} + w_{r+2}(-\mathbf{p})a(-\mathbf{p},r+2)e^{ip.x/\hbar}\right]\frac{d^3p}{\sqrt{2p_0}} \qquad (24)$$

in which (here and in what follows) $p_0 = +(\mathbf{p}^2+m^2c^2)^{1/2}$.

The bi-spinors are conveniently normalized to:

$$\bar{u}(p,r)u(p,s) = 2p_0/c\ \delta_{rs} = \bar{w}_r(\mathbf{p})w_s(\mathbf{p}) \qquad (25)$$

(in which a summation over spinor indeces is implied)[31]. The configuration space integral <.,.> takes the form:

$$\langle\psi,\psi'\rangle = \int\sum_r(\overline{a(\mathbf{p},r)}a'(\mathbf{p},r)+\overline{a(-\mathbf{p},r+2)}a'(-\mathbf{p},r+2))d^3p/cp_0 \qquad (26)$$

We may write the total Hamiltonian as:

$$\int\sum_r\left[p_0\overline{a(\mathbf{p},r)}a(\mathbf{p},r) - p_0\overline{a(-\mathbf{p},r+2)}a(-\mathbf{p},r+2)\right]d^3p/p_0 \qquad (27)$$

The definiteness, respectively indefiniteness, of <.,.> and the total energy is apparent. The quantized field may now be written in the form of Eq.(24), where we replace the a's by annihilation operators. In order that the ACR's Eq.(20) hold, it is necessary for the a's to obey the ACR's:

$$[\hat{a}^*(\mathbf{p},r),\hat{a}(\mathbf{p}',s)]_+ = p_0\delta^3(\mathbf{p}-\mathbf{p}')\delta_{rs} \quad r.s = 1,2,3,4. \qquad (28)$$

If we define:

$$N_r(\mathbf{p}) = \hat{a}^*(\mathbf{p},r)\hat{a}(\mathbf{p},r) \qquad (29)$$

The total number and energy operators take the physically transparent form:

[31] Our normalization is related to that in Schweber [1961] as follows: if $u'(p,r)$ are the bi-spinors defined by Schweber, then $u(p,r) = \sqrt{2m}\ u'(p,r)$. Likewise $a(p,r) = \sqrt{p_o}\ b(p,r)$, where b(p,r) is the annihilation operator defined by Schweber.

$$N = \int\Big[\sum_{r=1,2} N_r(\mathbf{p}) + \sum_{r=3,4} N_r(\mathbf{p})\Big]\, d^3p/cp_0 \tag{30}$$

$$\hat{H}_F = \int\Big[\sum_{r=1,2} cp_0 N_r(\mathbf{p}) - \sum_{r=3,4} cp_0 N_r(\mathbf{p})\Big]\, d^3p/cp_0. \tag{31}$$

The total number of electrons is the sum of the positive-frequency electrons and the negative-frequency electrons; the operator is positive definite. The total energy is the energy of all the positive frequency electrons, minus the energy of all the negative frequency electrons. The number operator is, of course, directly proportional to the charge:

$$Q = -eN \tag{32}$$

To summarize: we have a precise local equivalence between quantum field and 1-particle theory, and the QFT may be looked upon as the canonical second quantization of the 1-particle theory. All of the foregoing was implicit in the Heisenberg-Pauli theory of 1929.

The hole theory in second quantized form

We now consider the hole theory. The first attempts to cast this theory into second quantized form were due to Furry and Oppenheimer [1934][32], completed in December of 1933, and Fock [1933], using an earlier idea of Heisenberg in an application of the Jordan-Wigner formalism to atomic systems, in which vacancies in the closed shells were treated as positively charged particles. The definitive treatment was given by Heisenberg in [1934].

The basic idea is very simple. According to the hole theory, the *annihilation* of a negative-frequency electron is

[32] These authors did not proceed from the field theory, but (canonically) second quantized the 1-particle theory instead, then reinterpreted according to the hole theory. Because of the local correspondence between the (pre-hole) 1-particle theory and the quantum field theory, one rarely finds a logical distinction drawn between these methods. The quantum field theory of spin 1/2 system is still commonly presented in introductory texts as a canonical second quantization, then reinterpreted according to the hole theory; see, for example, Bjorken and Drell [1965], Schweber [1961].

interpreted as the *creation* of a hole in the negative energy sea. All that is needed is to replace the negative-frequency electron annihilation operator by a positive-frequency positron creation operator. In carrying out this substitution, we shall consider the removal of a negative energy electron with momentum **p** as equivalent to the creation of a positive frequency positron with momentum **-p**. We shall write[33]:

$$\hat{a}(\mathbf{p},r) = b(\mathbf{p},r) \qquad \hat{a}^*(\mathbf{p},r) = b^*(\mathbf{p},r)$$
$$\hat{a}(-\mathbf{p},r+2) = d^*(\mathbf{p},r) \qquad \hat{a}^*(-\mathbf{p},r+2) = d(\mathbf{p},r) \qquad r=1,2 \qquad (33)$$

The ACR's can be preserved under this substitution, because they are symmetric between the $\hat{a}$'s and $\hat{a}^*$'s. However the number operator for the negative frequency electrons is not equal to the number operator N^P for the positive frequency positrons, because:

$$N_{r+2}(-\mathbf{p}) = a^*(-\mathbf{p},r+2)a(-\mathbf{p},r+2) \neq d^*(\mathbf{p},r)d(\mathbf{p},r) = N^P_r(\mathbf{p})$$

We have rather:

$$a^*(-\mathbf{p},r+2)a(-\mathbf{p},r+2) = d(\mathbf{p},r)d^*(\mathbf{p},r) = K(\mathbf{p}) - N^P_r(\mathbf{p})$$

where $K(\mathbf{p})$ is an infinite constant. Writing $N^e_r(\mathbf{p})$ for the positive frequency electron number operator we now obtain in place of Eq.(30) and Eq.(31):

$$N = \int \sum_{r=1,2} \left[N^e_r(\mathbf{p}) + \left\{ K(\mathbf{p}) - N^P_r(\mathbf{p}) \right\} \right] d^3p/2p_0$$

$$\hat{H}_F \int \sum_{r=1,2} \left[p_0 N^e_r(\mathbf{p}) - p_0 \left\{ K(\mathbf{p}) - N^P_r(\mathbf{p}) \right\} \right] d^3p/2p_0.$$

In symbolic form, writing $N^{\pm}$, for the total numbers of positive and negative frequency electrons (note that $N^+ = N^e$):

$$N = N^+ + N^- \longrightarrow N^e - N^P + K_1$$

$$\hat{H}_F = \sum_{\mathbf{p}} \left[p_0 N^+(\mathbf{p}) - p_0 N^-(\mathbf{p}) \right] \longrightarrow \sum_{\mathbf{p}} \left[p_0 N^e(\mathbf{p}) + p_0 N^P(\mathbf{p}) \right] - K_2$$

or

$$\hat{H}_F = \hat{H}^+_F - \hat{H}^-_F \longrightarrow \hat{H}^e_F + \hat{H}^P_F - K_2$$

in which $\hat{H}^{\pm}_F = \Sigma_{\mathbf{p}} p_0 N^{\pm}(\mathbf{p})$. The first of these equations reads: the total number of negative frequency electrons equals the total number of negative frequency states minus the total

[33] There is no need to distinguish the b's and d's as operators from their c-number analogues. Consequently I ommit the hats.

number of unfilled states (holes or positrons). The second reads: the total negative energy of all the negative-frequency electrons equals the negative energy associated with all the negative frequency states minus the negative energy of all the unfilled states.

Normal ordering

Since the physical number and energy operators are, according to the hole theory, referred to the negative energy sea, (that is as differences in number and energy from their values for this "vacuum"), the infinite constants are to be discarded; in this way the spectra of the number and energy operators are actually changed, whereas before we had merely a new notation. For the Hamiltonian and number operator become simply:

$N = N^e - N^P$

$\hat{H}_F = \hat{H}^e_F + \hat{H}^P_F$.

The Hamiltonian now has a straightforward physical interpretation. The total number operator, however, is now indefinite; we multiply by -e and consider this operator the total *charge* of the field. The prescription, whereby these infinite constants are discarded, may be expressed as follows: all physically significant operators are to be *normal ordered*, that is, all creation operators are to placed to the left of all annihilation operators, with due account of the ACR's except that the RHS of all ACR's are to be put equal to zero. Thus:

$b_r(\mathbf{p})b^*_s(\mathbf{p})$ — normal-ordering $\longrightarrow -b^*_s(\mathbf{p})b_r(\mathbf{p}) = :b_r(\mathbf{p})b^*_s(\mathbf{p}):$

in particular:

$:N^-: = -N^P$, $:-H^-_F: = H^P_F$

$:N^+: = N^e$, $:H^+_F: = H^e_F$

where we have introduced the notation : :. It is crucial that the normal ordering is performed with respect to the new creation and annihilation operators, the b's, d's and their adjoints, because of course all expressions are already normally ordered with respect to the a's. The fields written in terms of the b's and d's, introducing the conventional notation

$$\left.\begin{aligned} w_r(p) &= u(p,r) = u((p^2+m^2c^2)^{1/2},p,r) \\ w_{r+2}(-p) &= v(p,r) = u(-(p^2+m^2c^2)^{1/2},-p,r) \end{aligned}\right\} r=1,2 \qquad (34)$$

become:

$$\hat{\psi}(x) = (2\pi\hbar)^{-3/2}\int_{\mathbb{P}^3}\sum_{r=1}^{2}\left[u(p,r)b(p,r)e^{-ip.x/\hbar} + v(p,r)d^*(p,r)e^{ip.x/\hbar}\right]\frac{d^3p}{\sqrt{2p_0}}$$

$$\hat{\psi}^*(x) = (2\pi\hbar)^{-3/2}\int_{\mathbb{P}^3}\sum_{r=1}^{2}\left[\overline{u(p,r)}b^*(p,r)e^{ip.x/\hbar} + \overline{v(p,r)}d(p,r)e^{-ip.x/\hbar}\right]\frac{d^3p}{\sqrt{2p_0}} \qquad (35)$$

We now have:

$$\begin{aligned} Q &= -e{:}\int \hat{\psi}^*(x)\hat{\psi}(x)d^3x{:} \\ \hat{H}_F &= {:}\int \hat{\psi}^*(x)H_0\hat{\psi}(x)d^3x{:} \end{aligned} \qquad (36)$$

Which can therefore both be considered integrals over corresponding local quantities, the charge density operator

$$Q(x) = -e{:}\hat{\psi}^*(x)\hat{\psi}(x){:} \qquad (37)$$

and the Hamiltonian density operator

$$\hat{\mathcal{H}}(x) = {:}\hat{\psi}^*(x)H_0\hat{\psi}(x){:}. \qquad (38)$$

However the new total number operator cannot be simply written as any normal ordered second quantized 1-particle expression (*i.e.* in the fields $\hat{\psi},\hat{\psi}^*$).

We may however write $\psi^e(x)$, $\psi^P(x)$ for the parts of the field ψ which refer to the electrons and positrons,

$$\hat{\psi}(x) = \psi^e(x) + \psi^P(x)$$

where:

$$\begin{aligned} \psi^e(x) &= (2\pi\hbar)^{-3/2}\int_{\mathbb{P}^3}\sum_{r=1,2} u(p,r)\, b(p,r)e^{-ip.x/\hbar}\,\frac{d^3p}{\sqrt{2p_0}} \\ \psi^P(x) &= (2\pi\hbar)^{-3/2}\int_{\mathbb{P}^3}\sum_{r=1,2} v(p,r)d^*(p,r)e^{ip.x/\hbar}\,\frac{d^3p}{\sqrt{2p_0}}. \end{aligned} \qquad (39)$$

Note that $\psi^e(x)$ is the spacetime electron *annihilation* field; $\psi^P(x)$ is the spacetime positron *creation* field. The total number operator may now be written:

$$N = N^e + N^P = \int \psi^{e*}(x)\psi^e(x)d^3x + \int \psi^P(x)\psi^{P*}(x)d^3x \qquad (40)$$

so that in this sense it may be regarded as the integral of the local density:

$$N(x) = \psi^{e*}(x)\psi^e(x) + \psi^P(x)\psi^{P*}(x) \qquad (41)$$

However the decomposition of the field $\hat{\psi}$ into positron and electron fields is a non-local operation on the fields, in the sense that as c-number expressions, one must know the

values of the field ψ over all space in order to determine the values of the electron and positron parts at any point in space.

Anti-commuation relations; causality

The spacetime commutation rules are:

$$\begin{aligned}
&[\hat{\psi}^*(\mathbf{x},t),\hat{\psi}(\mathbf{x}',t)]_+ = \delta^3(\mathbf{x}-\mathbf{x}')\\
&[\hat{\psi}(\mathbf{x},t),\hat{\psi}(\mathbf{x}',t)]_+ = [\hat{\psi}^*(\mathbf{x},t),\hat{\psi}^*(\mathbf{x}',t)]_+ = 0\\
&[\hat{\psi}^{e*}(\mathbf{x},t),\hat{\psi}^{e*}(\mathbf{x}',t)]_+ = [\hat{\psi}^{P*}(\mathbf{x},t),\hat{\psi}^{P*}(\mathbf{x}',t)]_+ = 0\\
&[\hat{\psi}^{e}(\mathbf{x},t),\hat{\psi}^{e}(\mathbf{x}',t)]_+ = [\hat{\psi}^{e}(\mathbf{x},t),\hat{\psi}^{e}(\mathbf{x}',t)]_+ = 0
\end{aligned} \tag{42}$$

To work out the remaining ACR's needs a little more work. We first note that[34]:

$$\begin{aligned}
\sum_{r=1,2} u(\mathbf{p},r)\otimes u\sim(\mathbf{p},r) &= \gamma^\mu p_\mu + mc\\
\sum_{r=1,2} v(\mathbf{p},r)\otimes v\sim(\mathbf{p},r) &= \gamma^\mu p_\mu - mc
\end{aligned} \tag{43}$$

where $u\sim = \bar{u}\gamma^0$ etc. With these we find:

$$[\hat{\psi}^e_i(x),\hat{\psi}^e_j\sim(x')]_+ =$$

$$(2\pi\hbar)^{-3}\iint \sum_{r,s=1,2} [\hat{b}(\mathbf{p},r),\hat{b}(\mathbf{p}',s)]_+ \, u_i(\mathbf{p},r)u_j\sim(\mathbf{p}',s)$$

$$\cdot\, e^{-i(p.x-p'.x')/\hbar} d^3p\,d^3p'/2p_0p_0'$$

$$= (2\pi\hbar)^{-3}\int (\gamma^\mu p_\mu + mc)_{ij} e^{-ip(x-x')/\hbar}\, d^3p/2p_0$$

$$= (2\pi\hbar)^{-3}(i\hbar\gamma^\mu\partial_\mu + mc)_{ij}\int e^{-ip(x-x')/\hbar}\, d^3p/2p_0$$

$$= i\,(i\hbar\gamma^\mu\partial_\mu + mc)_{ij}\Delta^+(x-x')$$

where we have written:

$$\Delta^+(x-x') = -i(2\pi\hbar)^{-3}\int e^{-ip(x-x')/\hbar}\, d^3p/2p_0 \tag{44}$$

This expression does *not* reduce to a δ function at equal times. A similar calculation for the ACR between the positron fields leads to:

$$[\hat{\psi}^P(x),\hat{\psi}^P\sim(x')]_- = i(i\hbar\gamma^\mu\partial_\mu + mc)_{ij}\,\Delta^-(x-x')$$

where

$$\Delta^-(x-x') = i(2\pi\hbar)^{-3}\int e^{ip(x-x')/\hbar}\, d^3p/2p_0 \tag{45}$$

Note that $\Delta^-(x) = \overline{\Delta^+(x)}$ and that, whilst neither vanish for

[34] See Schweber [1961 4e] Eq.(150), (154c) modified in accordance with our normalization.

spacelike x, nevertheless their sum:

$$\Delta(x) = \Delta^{+}(x) + \Delta^{-}(x) \qquad (46)$$

is the unique (up to a constant multiple) Lorentz covariant solution of the KG equation which vanishes for spacelike x. It is called the **causal commutator**. Fields which have ACR's (or CCR's) vanishing for spacelike separation are called **causal**. We also note that Δ has the property that:

$$\partial_0 \Delta(x)\big|_{x_0 = 0} = -\delta^3(\mathbf{x})$$

Therefore

$$[\hat{\psi}^{e}_{i}(x), \hat{\psi}^{e}_{j}{}^{\sim}(x')]_{+} + [\hat{\psi}^{P}(x), \hat{\psi}^{P}{}^{\sim}(x')]_{-} = i(i\hbar\gamma^{\mu}\partial_{\mu} + mc)_{ij}\Delta(x-x')$$

vanishes for spacelike (x-x'); by reference to the ACR's Eq.(39) we see that the LHS is just $[\hat{\psi}_i(x), \hat{\psi}^{\sim}_j(x')]_+$ so that

$$[\hat{\psi}_i(x), \hat{\psi}^{\sim}_j(x')]_+ = i(i\hbar\gamma^{\mu}\partial_{\mu} + mc)_{ij}\,\Delta(x-x'). \qquad (47)$$

Reworking the above for $t = t'$ one sees that the equal-time ACR reduces to $i(i\hbar\gamma^0\partial_0)_{ij}$ so that:

$$[\hat{\psi}_i(x), \hat{\psi}^{\sim}_j(x')]_+\big|_{t=t'} = \gamma^0\,\delta^3(\mathbf{x}-\mathbf{x}')$$

or

$$[\hat{\psi}_i(\mathbf{x}), \hat{\psi}^{*}_j(\mathbf{x}')]_+ = \delta^3(\mathbf{x}-\mathbf{x}')\delta_{ij}$$

consistent with the first of Eqs.(42) (and *a posteori* justifying the anticommutation relationships Eq.(28)).

Discussion

What is of central importance in the foregoing is that from a field-theoretic point of view, the extravagant assumptions of the Dirac hole theory are only necessary to obtain the correct plane wave expansion of the field, Eq.(35) - that is to represent the field as the sum of a *creation* field for the positron and an *annihilation* field for the electron - and to normal order the resulting field-theoretic expressions. Normal ordering amounts to the subtraction of c-number infinities; this practise is necessary even in the interacting NRQFT, and was familiar in heuristic discusions of the free NRQFT in connection with the zero-point energy[35]. By the mid 1930's no special physical interpretation was

[35] Note that there is no zero-point energy in the canonical second quantized theory or in the free NRQFT.

considered necessary for such subtractions[36].

Therefore the Dirac vacuum has this, and only this rôle: to justify the plane wave expansion. Over the next three decades reference to this structural input from the hole theory became less and less common; the plane wave expansion progressively came to be a postulate of the theory, which was to stand as logically independent of the hole theory. In this sense the hole theory, and in particular the negative energy sea, was no longer considered a literal description of the world, and at most played the rôle of a convenient heuristic (but see fn. 32).

Now consider the new features of the Lagrangian theory, supplemented by the correct plane wave expansion and normally ordered correspondingly, in comparison to the pre-hole theory expressions. The resulting theory I shall call the (spin 1/2) **standard formalism**; the assumption of the plane wave expansion provides a **particle interpretation** for the Lagrangian theory.

The total number operator $N^{+}+N^{-}$, a positive operator, has become the indefinite operator $N^{e}-N^{p}$, and is now (up to a constant) considered the total charge operator. Under the local correspondence of the NRQFT, the one-particle inner product locally corresponds to the charge operator - and the "naïve" probability density to the charge density operator. At the same time the 1-particle indefinite energy has become a positive operator. It is still possible to write the number and energy operators so as to preserve a local correspondence of sorts - by decomposing the fields into electron and positron creation and annihilation fields, and constructing the relavant operators as bilinear combinations

[36] There were exceptions. Pauli, for example, spoke contemptuously of "subtraction physics" and was critical of the hole theory on this score. He was particularily pleased that in the scalar field theory that he developed in collaboration with Weiskopf, such subtractions played a minor role. For this reason he dubbed it the "anti-Dirac" theory.

of these (*cf.* Eqs.(40), (41)) - but what is the Fock - space of positron states? The standard formalism appears to satisfy a local correspondence with the direct sum of *two* 1-particle theories, each with positive energy. Further the positron and electron creation and annihilation operators do not anticommute at spacelike separation and are defined in a non-local way by the standard (physical) fields. Do they obey the canonical relationships Eq.(35)(1.3.4)? If so is the inner product non-local? Finally, it is not hard to convince oneself that the number density operators defined by Eq.(41) are non-locally determined by the physical fields and do not anticommute at spacelike separation. In consideration of Eq.(56),(57) of (1.3.4) these features all seem to derive from a non-local inner product on the Fock space of states, if the creation and annihilation operators for electrons and positrons are canonically defined as in (1.3.4). We also note that the particle interpretation imposed preserves the ACR's for the physical fields, what would have been the ACR's for the creation and annihilation operators of the canonical theory with a local inner product. It is clear that to properly understand the relationship with the canonical second quantization we must define the Fock space explicitly, and if possible avoid the use of the point fields.

Pursuing these ideas will take us directly to Section 3.3 and 3.4. For the present I only wish to emphasize that the local correspondence that exists in the NRQFT is modified in drastic and ill-defined ways, and that in the absence of the hole theory one has essentially lost contact with the canonical second quantization of a 1-particle theory. With it Hilbert space theory has also largely dropped out of the picture; perturbation theory is conducted entirely in terms of momentum space integrals over the bi-spinors and the electron and positron annihilation and creation operators, the action of which on an underlying Hilbert space is never specified[37]. At the same time the negative frequency

[37] In standard text book accounts there is, I suggest, a tacit

solutions nowhere appear in the field-theoretic formalism. The Hamiltonian is positive definite, for both electron and positron; in *some sense* the positive-energy positron states are related to the negative-frequency electron states. A precise sense is defined by the hole theory, but if this is not a *literal* description of the world, one does not know what their *literal* relationship is. Indeed, one does not even know how to evaluate the claim that anti-particles are a *logical consequence* of RQFT, once the theory is cut off from the hole theory. Clearly the plane wave expansion implies anti-particles. But that is only to say that the particle interpretation implies the existence of antiparticles, a trivial claim. What is the basis of the particle interpretation, other than that it preserves the CCR's for the physical fields and makes physical sense?

The same considerations apply to the scalar field theory developed by Pauli and Weisskopf [1934]. The situation here is modified; there is no Hamiltonian form for the 1-particle theory, and there is no hole theory for the bosonic field; obviously the changes in sign for the energy and number operators come anyway from the anticommutators and would not arise from commutators. However, there is no apparent need for a hole-theoretic interpretation anyway. Let us see what happens in this case.

The charged scalar field

The Lagrangian density is now:

$$\mathcal{L} = \partial_\mu \overline{\phi(x)} \partial^\mu \phi(x) - \mu^2 \overline{\phi(x)} \phi(x) \qquad (48)$$

in which ϕ and $\overline{\phi}$ are to be treated as independent coordinates. The canonically conjugate variable to ϕ , respectively $\overline{\phi}$, is:

assumption that the Hilbert space (Fock space) of the electron and positron are both to be considered the Fock space over the free 1-particle positive frequency solution space of the Dirac equation. There is indeed no other Hilbert space in sight; from the point of view of Section 3.3 this is right in spirit if not in letter. From the point of view of Section 3.4, it is well off the mark.

$\partial\mathcal{L}/\partial\dot{\phi} = \pi(x) = \overline{\phi(x)}^{\cdot}$; $\partial\mathcal{L}/\partial\dot{\overline{\phi}} = \overline{\pi(x)} = \dot{\phi}(x)$.

The Hamiltonian then takes the form:

$$H = \int \left(\overline{\pi(x)}\pi(x) + \nabla\overline{\phi(x)}.\nabla\phi(x) + \mu^2\overline{\phi(x)}\phi(x)\right) \; d^3x \qquad (49)$$

Note that in contrast to the Dirac field, the Hamiltonian is *positive definite*. One obtains the conserved density

$$\rho(x) = i\hbar(\overline{\phi(x)}\dot{\phi}(x) - \overline{\phi(x)}^{\cdot}\,\phi(x)) \qquad (50)$$

either from the field equations or by Noether's theorem. What should be the inner product of the associated 1-particle theory:

$$\langle\phi,\phi'\rangle = i\hbar\int (\overline{\phi(x)}\phi'^{\cdot}(x) - \overline{\phi(x)}^{\cdot}\phi(x))d^3x \qquad (51)$$

is indefinite, as we have already seen. The situation is therefore the precise converse of the spin half field theory, where the energy is indefinite and the conserved density is positive definite. It is for these reasons that the modifications introduced by the hole theory are not required.

This system is quantized in the usual way:

$$[\hat{\pi}(\mathbf{x},t),\hat{\phi}(\mathbf{x}',t)] = -ih\,\delta^3(\mathbf{x}-\mathbf{x}')$$
$$[\hat{\pi}^*(\mathbf{x},t),\hat{\phi}^*(\mathbf{x}',t)] = -ih\,\delta^3(\mathbf{x}-\mathbf{x}')$$

which implies the CCR's:

$$\begin{aligned}[\partial_t\hat{\phi}^*(\mathbf{x},t),\hat{\phi}(\mathbf{x}',t)] &= -i\hbar\delta^3(\mathbf{x}-\mathbf{x}')\\ [\partial_t\hat{\phi}(\mathbf{x},t),\hat{\phi}^*(\mathbf{x}',t)] &= -i\hbar\delta^3(\mathbf{x}-\mathbf{x}')\end{aligned} \qquad (52)$$

What is *not* provided by the Lagrangian theory is the particle interpretation of the fields. This Pauli and Weisskopf put in by hand. They chose the plane wave expansion:

$$\begin{aligned}\hat{\phi}(x) &= (2\pi\hbar)^{-3/2}\int\left[a(\mathbf{p})e^{-ip.x/\hbar} + b^*(\mathbf{p})e^{ip.x/\hbar}\right]d^3p/\sqrt{2p_0}\\ \hat{\phi}^*(x) &= (2\pi\hbar)^{-3/2}\int\left[a^*(\mathbf{p})e^{ip.x/\hbar} + b(\mathbf{p})e^{-ip.x/\hbar}\right]d^3p/\sqrt{2p_0}\end{aligned} \qquad (53)$$

where the a's and a^*'s are respectively annihilation and creation operators for particles of charge e, and the b's and b^*'s are respectively annihilation and creation operators for antiparticles of charge -e. That this choice makes physical sense is established essentially by a number of consistency checks; that the CCR's of Eq.(52) are consistent with this choice, and that the total energy and charge (Eq.(50) multiplied by e) take on the expected form in terms of the number operators. Indeed these requirements

are met, as long as all the appropriate quantities are normal ordered. The normal ordering does not introduce any change of sign, however, in a theory using commutation reltionships, as remarked above.

Discussion

The 1-particle version of the foregoing is not in Hamiltonian form; in particular the negative energy difficulty is usually considered a consequence of the fact that the differential operator $i\hbar\partial_t$ applied to the negative frequency solutions has negative generalized eigenvalue. However the *expectation value* of $i\hbar\partial_t$ with respect to the sesquilinear form $\langle .,. \rangle$, for such states, is nevertheless positive, since $\langle .,. \rangle$ is negative for these states. In this way we see that there is no contradiction between the positivity of the Hamiltonian $\hat{H}_F$, where H_F is given by Eq.(49) above, and the 1-particle theory. On the other hand there is no 1-particle operator H *other than* $i\hbar\partial_t$ such that $\langle \phi, H\phi \rangle$ is positive definite. The superficial local correspondence with the 1-particle theory appears meaningless because the 1-particle theory does not make sense as a RQM, but only as a classical field theory. Because the plane wave expansion for the causal fields[38] involves creation *and* annihilation operators, it is anyway clear that the fundamental properties of the local correspondence cannot obtain, because operators of the form $\hat{\phi}^*(x)A(x)\hat{\phi}(x)$ (even when normal ordered) then include pair creation and annihilation terms.

The importance of the fact, that in both the spin 1/2 and scalar fields, the natural local bilinear forms in the fields lead to pair creation and annihilation effects, does not need stressing: it is for this reason that the

[38] As in the spin 1/2 theory, the spacetime particle and antiparticle creation and annihilation fields do not commute at spacelike separation. We also note that the total energy and charge, but not the total number operator, can be given as integrals over local energy and charge densities, in the sense that these are normal ordered bilinear forms in the causal fields.

relativistic quantum theory differs profoundly, both phenomenologically and mathematically, from the NRQM. For both theories we have now traced this circumstance to the specific property of the particle interpretation of the free fields, namely that the plane wave expansion includes *both* creation and annihilation operators. These particle interpretations do *not* follow from the canonical Lagrangian theory, nor - in the absence of a precise equivalence with any canonical second quantized theory - from the 1-particle theory. They are imposed purely in order to obtain a coherent physical interpretation for the free fields, in which negative energy states nowhere figure, and which makes sense in perturbation theory. We also note that the hole theory, in providing a bridge between the standard formalism and the canonical second quantization, introduces a new puzzle; why, when both the Dirac and KG equations admit negative energy states, does this bridge exist in the spin 1/2 case only?

It should also be noted that in all of this the charge conjugation operators play no interesting conceptual rôle. Kramers [1937] introduced this operator as follows. If one takes the adjoint of the Dirac equation in the presence of an external field

$$\left[\gamma^{\mu}(i\hbar\partial_{\mu} - \frac{e}{c}A_{\mu}(x)) - mc\right]\psi(x) = 0 \qquad (54)$$

one obtains the equation

$$\psi^{*}(x)\left[\gamma^{\mu *}(-i\hbar\partial_{\mu} - \frac{e}{c}A_{\mu}(x)) - mc\right] = 0$$

We recall that the γ^{i}'s are anti-hermitian and γ^{0} hermitian. From their defining relationship Eq.(5) it follows that

$$\gamma^{0}\gamma^{i*}\gamma^{0} = \gamma^{i} \qquad (55)$$

So inserting a factor $\gamma^{0}\gamma^{0}$ between ψ^{*} and γ^{μ}, and operating from the right by γ^{0}, one obtains:

$$\psi\sim(x)\left[\gamma^{\mu}\,(-i\hbar\partial_{\mu} - \frac{e}{c}A_{\mu}(x)) - mc\right] = 0$$

Taking the matrix transpose (denote ^{t}) we obtain:

$$\left[\gamma^{\mu t}(-i\hbar\partial_{\mu} - \frac{e}{c}A_{\mu}(x)) - mc\right]\psi\sim^{t} = 0 \qquad (56)$$

The equation is not yet in the form of Eq.(54), because of the transpose of the γ matrices. Suppose, however, that there exists a matrix $\mathcal{C}$ such that:

$$\mathcal{C}\gamma^{\mu t}\mathcal{C}^{-1} = -\gamma^{\mu} \qquad (57)$$

We may then insert a factor $\mathfrak{C}^{-1}\mathfrak{C}$ between the γ 's and the ψ, and operate from the left by $\mathfrak{C}$ to obtain

$$[\gamma^{\mu}(i\hbar\partial_{\mu} + \frac{e}{c}A_{\mu}(x)) - mc]\mathfrak{C}\psi^{-t} = 0. \tag{58}$$

We denote

$$\mathfrak{C}: \psi \longrightarrow \psi^{c} = \mathfrak{C}\psi^{-t} = \mathfrak{C}\gamma^{0t}\overline{\psi} \tag{59}$$

the (1-particle) **charge conjugation operator**. It is clearly anti-linear. The matrix $\mathfrak{C}$ exists[39] and can be chosen unitary; in the representation

$$\gamma^{0} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^{i} = \begin{pmatrix} 0 & -\sigma_{i} \\ \sigma_{i} & 0 \end{pmatrix}, \tag{60}$$

where the σ's are the Pauli spin matrices:

$$\sigma_{1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_{2} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_{3} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{61}$$

we choose:

$$\mathfrak{C} = = \begin{pmatrix} C & 0 \\ 0 & -C \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{62}$$

and Eq.(57) can be verified directly. $\mathfrak{C}$ is unitary so that $\mathfrak{C}$ is anti-unitary.

Eq.(58) is the Dirac equation for a particle of *positive* charge. If ψ is a positive frequency state then ψ^{c} is a negative frequency state, so that to obtain positive frequency solutions of the positive charge Dirac equation we must take the charge conjugate of negative frequency negative charge solutions.

All of this is already familiar from Dirac's discussion of charge conjugation in the scalar case. Clearly we can pass from the negative-frequency electron states to positive frequency positron states, interpreting ψ^{c} as a positron state. We may conjecture that the Fock space of the positrons is in fact the charge conjugate of the Fock space over the negative frequency solution space of the Dirac equation. But in fact this is a mistake; the charge conjugation in the field theory is given by precisely the same formal (anti-linear) map:

$$\hat{\psi} \longrightarrow \mathfrak{C}\gamma^{0t}\hat{\psi}^{*t} \tag{63}$$

[39] For a general existence proof see e.g. Schweber [1961 4h].

now applied to the quantum field. But by reference to the plane wave expansion and the action of the matrix $\mathfrak{C}$ on the momentum space bi-spinors, one finds that this operation is precisely equivalent to:

$$\begin{aligned} b(p,r) &\longrightarrow d(p,r) \quad b^*(p,r) \longrightarrow d^*(p,r) \\ d(p,r) &\longrightarrow b(p,r) \quad d^*(p,r) \longrightarrow b^*(p,r). \end{aligned} \tag{64}$$

In other words the Fock space of positive energy electrons is simply interchanged with the Fock space of the positive energy positrons. The two are in fact considered unitarily equivalent, and the charge conjugation operator in the field theory is a unitary operator[40]. This situation does not clarify the relationship between the field theory and the 1-particle theory, it has only given us another puzzle. In the scalar case the situation is exactly the same.

1.4.5. The Born interpretation.

These puzzles seemed to have been of interest to few physicists then or for that matter subsequently. They are no doubt puzzles which arise only when one tries to display the logical structure of RQFT in comparison to NRQFT[41]. The difficulties of implementing the Born interpretation in the relativistic theory are, however, another matter: here it is a question of extracting an empirical correspondence for the theory.

As we have seen the Dirac theory was discovered precisely by the attempt to find a relativistic theory, which yet

[40] Further, the 1-particle probability flux $\tilde{\psi}\gamma^{\mu}\psi$ (which in the field theory is interpreted as the charge-current density) does <u>not</u> change sign under $\mathfrak{C}$ - as is obvious when one considers the time component. See (3.3.8) for a complete statement of the problem, and (3.4.9) for its solution.

[41] One might think that mathematicians and constructive field theorists would have been concerned with these problems. Whilst Segal formulated the basic key to their solution (Section 3.4) he did not refer to them explicitly; nor has anyone else, to my knowledge. I find this situation remarkable. Of course, the free field theory is mathematically unambiguous, once one has defined an explicit Fock space (this is not hard to do, particularily in the scalar case, which has been the focus of most work in

preserved the fundamental properties of the NRQM; this seemed necessary to preserve its probabilistic interpretation. The Born interpretation seemed the one secure property of the new theory, with the inner product $\langle\psi,\psi'\rangle$ defined precisely as in the NRQM and first-order wave equation.

The first signs of trouble appeared in the peculiar properties of the "naïve" velocity operators, which one might think should be given by the (Heisenberg picture) operators precisely as in NRQM:

$\dot{x}_i = (i/\hbar)[H_0, x_i]$

where H_0 is the (free) Dirac Hamiltonian, Eq.(14):

$H = -i\hbar c\gamma^0\gamma^i\partial/\partial x^i + \gamma^0 mc^2$.

This velocity operator is just $c\gamma^0\gamma^i$, and clearly has a number of unphysical properties; first, its three components do not commute, and second, its spectrum is discrete, consisting of the points $\pm c$. Its derivation as above first appeared in Fock [1929], but it was first arrived at almost a year previously by Breit [1928], who became possessed of the idea that it should be possible to derive the Dirac equation from the classical relativistic Hamiltonian equation in much the same way as the KG equation is so derived under the replacements $p_\mu \longleftrightarrow i\hbar\partial_\mu$. He considered the Hamiltonian equation in the form:

$E/c + eV/c = mc(1-v^2/c^2)^{1/2} + \mathbf{v}/c\ (\mathbf{p} + e\mathbf{A}/c)$

(in which $\mathbf{v}$ is the particle velocity) and noticed that if one makes the substitutions:

$(1-v^2/c^2)^{1/2} \longrightarrow \gamma^0$; $v^i \longrightarrow \gamma^0\gamma^i$

$\mathbf{p} \longrightarrow -i\hbar\nabla$; $H \longrightarrow i\hbar\partial_t$

then the Dirac equation is obtained. He therefore tried to argue *a priori* that the eigenvalues of v_i/c and $(1-v^2/c^2)^{1/2}$

constructive field theory) but its relationship to the 1-particle theory, to the NRQFT, and to NRQM, does not appear to have been systematically studied. The view that there simply is no precise relationship between these theories, that the relativistic theory differs in some fundamental but not clearly defined sense from the NRQFT, is false, because Segal (albeit implicitly) proved otherwise long ago, in the mid 1960's.

must be ±1, and that one should not expect that the components of the velocity be simultaneously measurable. In particular he appealed to a disturbance theory of measurement[42] to deduce that unlimited certainty in velocity requires that the electron have an "effective" infinite mass, which formally follows if its velocity is of magnitude c.

The arguments of Breit set, as it were, the tone of the debate; the argumentation was highly formal, made much use of the time-energy uncertainty relationships (the most subtle and difficult to formulate in a consistent way), and invariably appealed to a disturbance theory of measurement. As another example, Fock in his [1929] paper proposed the quantity $\mathbf{p}/m\nu$, where $\nu = d\tau/dt$ and τ is the proper time; this quantity is classically equal to the velocity and goes over to a triple of commuting operators, with continuous spectra in the interval (c,-c). There were, however, difficulties in the applications of this operator in the Dirac theory. Fock proposed a distinction between "mechanical velocity" (given as above, and referring to the particle picture) and "field velocity" (referring to the wave picture). The latter were to be given by the $\gamma^0\gamma^1$ operators.

With Schrödinger's work the following year [1930] a new physical picture was elaborated that remains a popular heuristic to this day. Considering the velocity operators as given by $\gamma^0\gamma^1$ he argued that they could be represented as the sum of two terms, one of which (similar to Fock's "mechanical velocity") could be interpreted as a "mean" velocity, and the other of which has an oscillatory time-dependence with magnitude c. From this stemmed the concept of "Zitterbewegung", a rapid velocity fluctuation about a mean velocity, soon to be interpreted as an

[42] Then much in vogue; this stood at the heart of Bohr's interpretation advanced at the Solvay conference of 1927, and in the recent Heisenberg theory of the uncertainty relationships [1927].

inteference effect arising from virtual pair creation and annihilation processes.

Schrödinger did not at the time accept that such transitions were physically meaningful, and there followed a series of papers in which he tried to modify the Dirac theory so as to prohibit them (see Schrödinger [1932] for a summary). In the course of his work he made an important contribution: he introduced the distinction between *odd* and *even* operators, *i.e.* between operators which did, and did not, induce transitions between positive and negative frequency solutions of the Dirac equation. In particular the momentum operator is even, whilst the naïve position operator $(q_i\psi)(\mathbf{x}) = x_i\psi(\mathbf{x})$ is odd. Thereby his previous decomposition of the naïve velocity operator into two parts could be understood as its representation as the sum of an even and an odd part.

In the context of the hole theory, transitions from positive to negative frequency states are to be interpreted as pair creation and annihilation events; therefore the occurrence of odd operators in the theory necessitates the switch from a 1-particle theory to a many-particle theory. Moreover an interpretation as to *why* the naïve position operator is odd follows immediately from the application of the time-energy uncertainty relationships to the disturbance theory of measurement. An observation of the position of an electron to within the Compton wavelength $\hbar/mc$ implies uncertainty in the energy of the order of the rest mass energy, and hence sufficient to produce pair creation. One will not thereby obtain a localized *one*-particle state.

This conclusion follows from the usual thought experiment, the γ-ray microscope. The Abbe theory of resolution leads to the conclusion that, in the ideal case that all the forward-scattered light is collected by the objective, a point particle will be resolved along a given coordinate axis x_i in the focal plane to an accuracy $\Delta x_i \approx \lambda$, where λ is the wavelength of the *scattered* radiation. λ cannot be made

arbitrarily small; in the limit of arbitrarily high incident frequencies the forward-scattering, proceeding via the Compton effect, are of order E/h, where E is the particle energy prior to the scattering[43]. Consequently $\Delta x_i \approx hc/E$, which reduces to the Compton wavelength when the particle is stationary, $E = mc^2$. In the limit $c \longrightarrow \infty$ the position coordinates can be determined with arbitrarily high accuracy according to this argument. This fact was taken by Pauli in his influential handbuch article (Pauli [1933a]) as the fundamental justification for the configuration space Born interpretation in NRQM. By implication, only if particle energies are arbitrarily large[44] may the position of a particle be determined to arbitrarily high accuaracy; but if an upper value is set to exclude the possibility of pair production, an absolute restriction on the possibility of position measurement by this experiment results.

But elsewhere in the same article he took a different line[45]:

> Whether the limit ($\Delta x > h/mc$) ...has a significance or whether it can be rached by indiirect methods cannot be decided beforehand by elementary considerations. It is completely dependent on the foundation on which a relativistic quantum mechanics can be successfully built up. (Pauli [1933a]).

It was evident to everyone who touched on relativistic measurement theory that the Bohr-Heisenberg measurement theory of NRQM was wholey inadequate to the epistemological problems of RQT. The most ambitious and far-ranging analysis was undertaken by Landau and Peierls [1931]; they too concluded that particle localization is impossible to regions smaller than the Compton wavelength, but from a more general basis. In contrast to Pauli's later distinction (Pauli [1933a]) between measurements of the first and second

[43] this theory is semi-classical, in the manner of all such thought experiments.

[44] This was the basis of Breit's argument; only in the limit $E \longrightarrow \infty$ can each x_i have a precise value. But in this limit the velocity must be c.

[45] This is the point of view taken here.

kind, they considered the essential requirement of a measurement process be that the result is meaningfully *retrodicted* of the observed system. They concluded that a position measurement must be made over a time interval h/E if the result is to refer to the position of the particle prior to the measurement, and that the position is then determined only with respect to this time interval. Since the change in the velocity over a time interval Δt is limited in general only by c, one can retrodict only to an accuracy of $c\Delta t = ch/E$. The authors pointedly expressed their pessimism concerning the applicability of the wave mechanics to the relativistic domain.

As is well known, their analysis of the measurability of the electromagnetic field quantities motivated Bohr and Rosenfeld's labyrinthal treatise on this question (Bohr and Rosenfeld [1933]). Bohr was concerned to prove that all commuting operators could be simultaneously measured according to the semi-classical models. When Landau and Peierls concluded from models of this type using point charges, that not all the components of the electric and magnetic fields could be simultaneously measured, Bohr promptly introduced extended test bodies, arguing that the atomicity or otherwise of charged matter lay outside the scope of the QEM field theory. The unsatisfactory features of this strategy evidently troubled him; in his subsequent attempt, more than 20 years later, to consider the same problem in the context of a fully interacting QED (Bohr and Rosenfeld [1950]), the ambiguities of both theory and philosophical strategy are I believe painfully obvious.

In this period only Furry and Oppenheimer perceived the central thrust of Schrödinger's analysis; that one should look for operators which in some sense are associated with the particle position, but which are *even*. At least one must be able to define an "approximate" particle localization. Schrödinger had constructed just such an triple, the set:
$q_i = q_i + i\hbar c/2\ (\gamma_i H_0^{-1} + cp_i H_0^{-2})$
with H_0 the free Dirac Hamiltonian. Furry and Oppenheimer

investigated this operator in the context of the hole theory, with some limited success[46]. They found positive frequency generalized eigenfunctions f_q (with zero dispersion) but were puzzled by the apparent ambiguity, that whereas the probability of finding a positive frequency electron in the range dq is $|\langle f_q, \psi\rangle|^2 dq = P dq$, this probability is not equal to the probability $|\langle \delta_q, \psi\rangle|^2 dq = W dq$ (where δ_q is defined on $L^2(M, d\mu)$ as in (1.3.4))[47]. They continued:

> $P dq$ depends on the value of ψ in a region of the order h/mc about q. When nothing is known about the state of the electron, ψ, before the measurement, the value of the coordinate x is fixed by the determination of q only within the limit of the Compton wavelength. When it is known that the kinetic energy of the electron before the observation is certainly as great as E, then the probability of observing a given value for q depends on the magnitude of the wave function ψ only throughout the region $q \pm hc/E$; and the position of the electron before the experiment can be inferred from the results of the experiment with a correspondingly greater precision. One may say therefore that, whereas it is possible by experiment to localize an electron, in the sense that the position of the electron after the experiment may be determined as precisely as one wishes, it is nevertheless not possible to determine the position of an electron in an arbitrary state with a position greater than h/mc. It will be seen that this corresponds exactly to the possibilities offered by the gamma-ray microscope where the intial determination of position is necessarily unprecise by about hc/E, but where a second determination of position, after the electron is known to have scattered a hard gamma-ray, may in principle be made precise. (Furry and Oppenheimer [1934 p.248]).

This passage is a convoluted attempt to integrate their own results, which seemed to show that it was indeed possible to define dispersion-free generalized eigenstates in the q's, with the analysis of Landau and Peierls, and the Abbe theory of resolution: a messy combination. The situation appeared

[46] They did not comment on the fact that the q's are not a commuting triple.

[47] Apart from the difficulty that the operator triple q_1 is non-commuting, there was no reason for Furry and Oppenheimer to be confused at this point. They were essentially on the right track (see (3.2.7); there is no reason for $P\,dq$ to equal $W\,dq$, and the latter is just the naive probability distribution.

so confused that interest soon shifted to other things; one did not seem to be led terribly astray thinking of the naïve position operators, and the configuration space wave functions, as having at least something to do with particle locality; and that was good enough to do perturbation theory with. And for the experimentalists, of course the curvature of particle trajectories was enough to calculate the associated momentum - as though the ionization trails were produced by classical particles.

The empirical support of QED (scattering theory) was thus based on the correspondence principle for the first three decades of its existence. It must be emphasised, that from a logical point of view, failing a satisfactory and precise definition of the position operators *one does not have an approximate definition of particle localization either.* There is *no* theoretical justification for the practise of the experimentalists, except through the correspondence principle.

This claim is somewhat controversial; undoubtedly the momentum space Born interpretation is valid in the relativistic theory. But I claim that what is measured in the laboratory is *always* the spacetime correlations of ionization phenomena. In the view taken here, measurement theory is (or should be) concerned with the behaviour of microphysical systems in metastable macroscopic environments - be they photons in interaction with photographic emulsion, or electrons in a superheated fluid. The Born interpretation correlates properties of the microphysical system to phase transitions in a small domain of this macroscopic environment - and it is fortuitous that a simple correlation of this kind is successful. It is, however, another matter altogether to hypostesize correlations between relatively high level theoretical properties of a a microphysical system (momentum, for example), with high order theoretical properties of a measurement apparatus (the computer print-out with the "measured" values of momentum written on it). What is immediately observed, and from which the

"measured" momentum is deduced, are, I repeat, spacetime correlations of ionization phenomena.

There is one last line of defence; and that is that momentum may be measured using the Doppler effect. This is no defence at all, because the problem is transferred to the relationship of the scattered radiation to observable events. One then has to deal with the correlation of spacetime properties of the radiation field to a macroscopic environment, *viz* the photographic emulsion or, for that matter, the human retina. I raise this issue more in deference to Dirac, one of the small handful of physicists of the 30's still capable of independent thought concerning the general interpretation of the quantum theory. It will have been noticed that he played no rôle in the translation of the hole theory into field theoretic terms, nor did he contribute significantly to RQFT *per se*. More and more, his efforts were subversive, and I suggest this reflected a deep disatisfaction with the logical and epsitemological structure of the theory. I defend this viewpoint on the basis of his paper "Relativistic quantum mechanics", written in early 1932, where he opposed the Pauli-Heisenberg theory of interacting electromagnetic and electron fields, and developed some ideas which he later incorporated into a systematic theory of QED in which the electromagnetic field did not appear to be treated as a dynamic system. This later theory was published in collaboration with Podolsky and Fock (Dirac *et al* [1933]), but the equivalence there established with the Heisenberg-Pauli theory at once undermined the new theory: it proved no more than a way of developing the interaction picture, without explicitly introducing the free field theory. This theory has become subsequently known as the Dirac "multiple" time formalism, and it is supposed to have something to do with proving manifest Lorentz covariance; its motivation was nothing of the kind, but stemmed from the earlier paper. The key passage is from the introduction:

> An attempt at a comprehensive theory on these lines has been made by Heisenberg and Pauli. These authors regard the field itself as a dynamical system amenable to

> Hamiltonian treatment and its interaction with the particles as describable by an interaction energy, so that the usual methods of Hamiltonian quantum mechanics may be applied. There are serious objections to these views, apart from the purely mathematical difficulties to which they lead. If we wish to make an observation on a system of interacting particles, the only effective method of procedure is to subject them to a field of electromagnetic radiation and see how they react. *The very nature of an observaton requires an interplay between the field and the particles.* We cannot therefore suppose the field to be a dynamical system on the same footing as the particles and thus something to be observed in the same way as the particles. The field should appear in the theory as something more elementary and fundamental (Dirac [1932 p.454])[48].

New ideas were needed, not only in the measuement problem, but in the general context of interpretation. The RQFT was, I suggest, an *ad hoc* formalism, with no straightforward connection with the canonical structure of NRQFT, whilst RQM scarcely existed as a coherent mathematical or physical theory, even in the free case.

1.4.6. The abstract method.

In this situation, from a philosophical if not from a physical point of view, the central advances in the early 1930's came from a small handful of mathematicians and physicists - from Hermann Weyl, Pascal Jordan, Paul Dirac, Eugene Wigner, and above all John von Neumann. I shall discuss all these contributions at great length in Part 2, and the pioneering work of Wigner (at the initiation of Dirac) in Section 3.1. For the moment I wish only to comment on three results which, I believe, mark the conscious beginnings[49] of an altogether new relationship between

[48] Of course, this view is a remarkable about-turn; it was Dirac who first treated the electromagnetic field as a Hamiltonian system.

[49] The idea of interpreting NRQM according to a non-classical logic was first proposed by Zawirski [1932]. However, neither this paper, nor the later discusions of Zwicky [1933], and Margenau [1934], (see Jammer [1974 8.1]) properly implement abstract methods, no more than does the reduction of scientific propositions to Ramsey sentences constitute an abstract formulation of science in the sense

mathematics and physical theory, which I call the abstract expression of physical law or, more briefly, the abstract method. These results are theorems, namely: Wigner's [1931] theorem on the unitary or antiunitary nature of a linear transformation on Hilbert space which leaves the transition amplitudes invariant, the Stone-von Neumann theorem on the uniqueness (up to unitary equivalence) of the CCR's in Weyl form (von Neumann [1931]), and von Neumann's [1932] proof of the impossibility of hidden variable theories for quantum mechanics.

The uniqueness theorem was based on recent results of Stone conerning the relationship between unitary representations of the additive group on the reals, and projection-valued measures on Hilbert space. We shall explore these relationships at length in Section 2.4, and for the moment I only wish to point out that one essentially replaces what Born called "the strange equation" $[p,q] = -i\hbar$ with something much more transparent: the requirement that the mathematical structure which models a quantum system be *invariant under the group of rigid motions* (Euclidean group).

Wigner realized that if a symmetry transformation leaves invariant the *transition amplitudes* of the form $|\langle f,g\rangle|^2$ (for all f,g in a Hilbert space $\mathcal{H}$) then this transformation must be given by a unitary or anti-unitary operator on $\mathcal{H}$. It was implicit in this result that it is the *rays* of Hilbert space that are physically significant, rather than vectors, wave-functions, or "waves" - however intuitively appealing the latter (essentially classical) interpretation. Developing this insight, one is led to consider a symmetry transformation as a map on Hilbert space which preserves the

developed by von Neumann et al. The former are interpretations of conceptual and mathematical theories of the world. The latter are intended to constitute the conceptual and mathematical description of the world.
The very first beginnings of the abstract method are rather to be found in Heisenberg [1925], as I made clear in (1.1.1).

structure of closed sub-spaces of Hilbert space, equivalently the *projective geometry* of Hilbert space. unitary transformations become abstractly characterized: they are *automorphisms* on a projective geometry[50].

With the von Neumann "no-go" theorem one had something much more ambitious. I do not wish to consider the context of von Neumann's interest in hidden variable theories; it is sufficient for our purposes that he wished to prove a general property of quantum mechanical probability, namely that the expectation value of an observable R, represented as as operator on a Hilbert space, must be given by $\langle R \rangle = Tr(\rho R)$, where ρ is a density matrix. It was then a simple step to show that there is no ρ such that $\langle\ \rangle$ is dispersion free for all R. He proceeded by making the following postulates concerning the expectation value and the nature of the correspondence between operators on Hilbert space and "quantities" that pertain to physical systems:

I. If a quantity is represented by the operator R, then a function f of this quantity is represented by the operator f(R).

II. If quantities are represented by the operators R,S,..., then the sum of these quantities is represented by the operator R+S+..., regardless of whether the operators commute or not.

III. If the quantity R is by nature nonnegative, then its expectation value $\langle R \rangle$ is nonnegative.

IV. If R,S,... are arbitrary quantities and a,b,..., real numbers, then $\langle aR + bS + ... \rangle = a\langle R \rangle + b\langle S \rangle + ...$

Von Neumann's assumptions proceeded from the basis that "quantities" were to be represented as operators on a Hilbert space. Jordan generalized this strategy still further; he *gave up* this concrete representation. By 1932 when he published his first paper on the series, "Uber eine

[50] I do not wish to suggest that this insight was formulated on the basis of Wigner's theorem; as we shall see in (2.1.2) von Neumann was led to this perspective by a different route.

Klasse nich assoziativer-hyperkomplexen Algebren", he had long since concluded that the self-energy problems of QED require "radically new, sweeping ideas...which cannot be evolved from the correspondingly calculated Maxwell theory: quantum mechanics will have to find some way of developing itself" (Jordan [1929]). In [1932] he also drew attention to the paper of Landau and Peierls [1931].

In giving up the concrete representation of operators on a Hilbert space, Jordan was essentially returning to the situation of 1925; to the Heisenberg "collections of quantities" and to Dirac's "q-numbers" - abstract objects, algebraically defined. We shall take this up at the beginning of Part 2.

1.4.7. Some questions of Ehrenfest.

In the same year Paul Ehrenfest published a short note, entitled "Einige die Quantenmechanik betreffende Erkundigungsfragen", which he began with the words:

> Let it be permitted in the following, to pose a set of questions, which must have arisen in similar fashion with almost every professor who has ever had the misfortune to to present an introduction to quantum mechanics to an audience which is trained to think critically. Of course, these questions can, especially in the way they have been formulated here, be put aside as "futile" if one wants to make life easy for oneself. Appearances and propriety even demand this. Now, therefore, somebody will have to take the odium[51] upon himself, to pose them nevertheless, in trust that there still exist a few scientists who possess the art to give sensible answers to senseless questions, and what is more, in a clear and simple manner. (Ehrenfest [1932 p.555]).

Ehrenfest was in a despairing mood, and shortly to take his own life[52]. He posed three groups of questions: the first

[51] Odium, a rare word, even more so in German than in English: disapproval, possibly ridicule from those in authority.

[52] Pauli was gracious enough to attempt to answer Ehrenfest in print (Pauli [1933b]); I do not consider that any of Pauli's comments get to the heart of the issues raised by Ehrenfest, who all his life was the most honest and penetrating of critics. If the resolution offered here is correct, nor

concerned the rôle of *complex numbers* in quantum mechanics, the second the *local relationship* between the classical fields and the configuration space wave functions, and the third the rôle of *spinors* in quantum mechanics, and the *Dirac bi-spinor* in relativistic quantum theory. Concerning the first he noted that whilst in classical physics one invariably deals with *real-valued* quantities, nevertheless one may introduce complex numbers for the purpose of economy - for example to form the vector

$\mathbf{M} = \mathbf{H} + i\mathbf{E}$

where H and E and the magnetic and electric fields, and in this way express the Maxwell equations in the form:

$-i/c\ \partial_t \mathbf{M} = \nabla \times \mathbf{M}\ ,\quad \nabla.\mathbf{M} = i\rho$

But the same physics may always be expressed by the use of purely real equations. Not so with the quantum theory. Why is this?

This question may be considered a central objective of the abstract approach: it is not fully clarified in this thesis, however. A partial answer is provided in the theory of Section 2.2, 2.3, and 3.4. At the same time the general tenor of the abstract approach as I shall formulate it, is that it permits a realist interpretation of a structure which may be *represented* by concrete mathematical objects, fragments of which do not themselves "correspond" to the real world in the manner of classical physics. It is only their structure which corresponds to the structure of the world; complex numbers in Hilbert space theory will be introduced in terms of a real linear transformation on the classical phase space (Section 2.5, 3.4).

could Pauli have made any real headway; we stand on the shoulders of giants, on the lifetime work of von Neumann, Wigner, Segal and Mackey, and many others. However there is one insight that was already available to Pauli, and which he did not exploit: in consideration of Ehrenfest's query as to the role of complex numbers in quantum physics, it was implicit in Wigner's [1931] theorem that if it is rays, and not wave-functions, that are fundamental, then in some sense it is a gauge group and not complex numbers per se which is indispensible. See below and Section 3.4.

Ehrenfest's second question is broadly addressed to the relationship between the classical fields and their relationship, on the one hand to the 1-particle theories, and on the other to the associated quantum field theories. His queries were qualitative, but there was one specific issue: he expressed bewilderment at the Hamiltonian form of the scalar theory proposed by Landau and Peierls [1931]; namely

$i\hbar\partial_t\psi = (-\hbar^2\Delta + m^2c^4)^{1/2}\psi.$

This equation, Ehrenfest complained, is clearly non-local in some sense. Why is this so? And why is there no Born interpretation applicable to configuration space states?

I have already elaborated these issues at length in (1.4.4). In the spirit of Ehrenfest, let me now summarize the unsatisfactory features of this relationship as of the mid 1930's; they are, of course, inter-related.

1. What is the 1-particle theory? What is the 1-antiparticle theory? Are the relativistic wave equations to have a dual rôle, on the one hand as classical field equations, and on the other hand as the Schrödinger equations of a 1-particle (or antiparticle) theory?

2. If so surely both particle and antiparticle states should have positive energy; how is this possible? And how can the KG equation be interpreted as a Schrödinger equation?

3. What is the Fock space of the quantum field theory? Why *must* the fields be linear combinations of creation and annihilation operators? In free field theory no creation or annihilation processes are involved; can free-field theory be formulated as a canonically second quantized particle-antiparticle theory? Can a local correspondence be defined in this case?

4. Is a Born interpretation possible in RQT in configuration space? Is this difficulty related to other peculiar features of RQT?

5. Why is a reinterpretation of negative energy states as positive energy antiparticle states possible for fermion fields (hole theory), but not for boson fields? Is such a

reinterpretation even necessary for boson fields? Yet does not the KG equation also admit negative energy solutions?

All of these problems are completely resolved in Part 3, most particularly in Section 3.4.

Concerning Ehrenfest's last set of questions (on the spinor formalism), these are properly formulated in the context of the relationship between group theory and quantum theory; their resolution is to be sought in the fact that Wigner's theorem permits projective rather than true group representations, and hence ultimately in the fact that the intrinsic structure of physical systems is modeled by a lattice or abstract algebra. For the explicit treatment of projective representations (spinors) see (3.1.5); for the interpretation of the Dirac bi-spinors in terms of spin-bundle representations, see (3.3.6).

The interpretation of quantum theory developed in Part 2 is a realist interpretation; as such we must develop an account of the measurement process. There is one, fundamentally new feature of QFT[53] which has been exploited to this end; namely the existence of inequivalent representations. The general interpretation of non-Fock representations is explored in Section 3.5 and in (3.5.5), (3.5.6) we review the so-called Hepp theory of measurement.

[53] RQFT offers new dynamical perspectives, but these ideas (particle creation and annihilation) mesh easily enough with classical intuitions, if not with classical theory. Cushing [1988] has recently argued that all of the really paradoxical features of quantum theory, including RQFT, are already present in NRQM. I will argue that this is false, and the most dramatic evidence that it is false is provided by exploiting just those fundamentally novel features of QFT in application to the measurement problem.

PART TWO: MATHEMATICAL AND PHILOSOPHICAL FOUNDATIONS

Introduction: on abstract methods

The rôle of abstract methods in physics is not easily defined; equivalent terms include axiomatic, constructive, or even operational methods, and it is obvious that - if only on the grounds of terminology - a variety of distinct philosophical motivations are involved. Let me then summarize the most popular positions[1].

1. **Constructive methods**. This is the most neutral term. Generally, it means no more than that the mathematical theory is developed in a rigorous way. Of course standards of rigor vary enormously; the gap between even the most schematic constructive treatment and the largely figurative mathematics used by physicists is, however, wider still.

2. **Axiomatic methods**. This term is more widely used most particularly in the context of quantum field theory and the program, begun in the mid 1950's by the likes of Friedrichs, Wightman, Segal, Haag, Borchers, and Ruelle, to develop a mathematically consistent non-trivial relativistic quantum field theory. Since everyone knew (from the success of QED) what such a theory should look like, these people began with a list of properties ("axioms") which such a theory should possess and then sought to either prove them inconsistent or else construct a consistent model (or class of models) which has these properties. As a result attention is focused on abstracting from QED and NRQT the most fundamental and general properties.

[1] It would be misleading to suggest that a publication which elaborates, for example, an axiomatization of quantum mechanics, is thereby committed to a philosophical approach to physical theory (i.e. an axiomatic approach).

3. **Operational methods**. In contrast to the above positions there is a very extensive literature on the philosophy underlying the use of this term, or rather the philosophies, because different authors mean very diferent things by it. In his essay "The Mathematical Meaning of Operationalism in Quantum Mechanics", Irving Segal said that "An operational treatment may be described as one that deals exclusively with observables[2]" (Segal [1959b]). But a more common view is that the operational approach is characterized by its focus, at the level of postulates or axioms, on what can be done in the laboratory. In this spirit Jauch is concerned to provide an operational meaning (in the traditional positivist sense) of conjunction and disjunction in the propositional lattice approach, and Haag and Kastler believed that in Fell's notion of weak equivalence one had a characterization of equivalence which fitted better with the limitations of laboratory measurements than that of unitary equivalence, and which should therefore play a central rôle in the logical structure of quantum theory (see below).

This approach might also be called *phenomenological*, in that one is almost always concerned to interpret the fundamental postulates of the theory in terms of strictly observable laboratory phenomena.

The operational approach obviously differs from the others in that it is not explicitly motivated by a concern for rigor and mathematical insight; nevertheless one is driven to similar mathematical techniques . A better way of putting it is that operationalism is very useful in all kinds of mathematical physics (the formal and probably inconsistent excesses of S-Matrix theory as well as the rigorous conservatism of foundational studies in NRQM) and that one is driven to a rigorous and general mathematical approach in foundational studies in physics whatever one's philosophy. There are a few exceptions - in particular the Bell inequality has provided a really very profound insight into

[2] It is clear from the context that Segal had the technical notion of "observable" in mind.

the nature of quantum theory using elementary methods - but I think they are exceptions which prove the rule: there has been very little progress in advancing our understanding of physics , particularly quantum physics, within natural language and elementary mathematics.

I am not quite sure why this should be so; perhaps it is a reflection of the ongoing mathematization of physics, which is to say it has the same source as whatever it is that makes mathematics so successful in physics. But this, I suspect, is to make of it too mysterious a problem, when one might say instead that mathematics is the precise study of structure and that the foundational study of physical theory is precisely concerned with logical structure.

That is roughly the point of view taken here. I do not want to make too much of it because the fact that dynamics has such a beautiful and complex structure is presumably connected to the fact that formal mathematical equations are so unreasonably effective in the physical sciences (to paraphrase the title of Wigner's paper on this topic). For example: Wigner in this paper drew attention to the popular example of an inverse square law. Whilst this law was initially published on the basis of an empirical accuracy of around 16% it has been found to be accurate to something like one part in 10^7 for suitable systems. A paradigm case? Yet with the emphasis on structure, rather than formalism, and in particular from the viewpoint of Gauss's law one sees that the exponent must be integral if the dimensionality of space is integral and it must be two if space is 3-dimensional. This early structural relationship between force laws and geometry would not have been possible with an exponent of 1.999999.

It is plausible already that the analysis of structure should be a more abstract business than the formal theories of physics, but there are two further features that I would like to point out. First, failing a single comprehensive dynamical theory one has a fairly large variety of elementary particles and their associated fields; one also

has a variety of topological groups which enter into physics in a similar way (and not only the two space-time groups), and one has the general methodology that it is helpful to study physically unrealistic models as a first step. All of this leads one to look for the very general properties shared by whole classes of theories and it seems inevitable that these properties will be rather abstract. So one ends up using rather abstract ideas like that of a C^*-algebra or a lattice or a system of imprimitivity and these can look purely mathematical to the uninitiated. Second - and this is a rather vague idea - we are all accustomed to accepting without question that NRQM consists of a small collection of laws expressed in terms of formal equations like the Schrödinger equation and explicitly given operators on a concrete Hilbert space and it is not so common to specify the theory in terms of an abstract Weyl algebra and the action of the inhomogeneous Galilean group on this algebra. There is perhaps the feeling that the former is physics and the latter is mathematics; it is certainly true that the latter requires a different and perhaps broader mathematical background but once one has understood Wigner's theorem on the representations of this sort of group and the Stone-von Neumann-Mackey theorem on the representations of Weyl algebras for systems of finite degrees of freedom one sees that one gains nothing by all this concrete formalism that is not already contained in the abstract description , which is much easier to make rigorous and in which one has such a very clear separation of spacetime structure and quantum system. It is true that it is very nice to point to some string of symbols and say *that* is a law of nature but these symbols have no existence, whereas the mathematical structure which underpins them may in a sense exist, both in the world (in the sense that geometry may be manifested by phenomena) and in the mind (in the sense that one can grasp the meaning of the infinitesimal action of a group better than one can grasp the meaning of a differential equation, when one abstains from specific phenomenological applications).

Working physicists will of course do nothing of the kind.

They will know each term in a differential equation in a direct phenomenological way, that is in terms of how modifying this term the phenomenology predicted will be different. This is a kind of insight that is on the whole denied to both the mathematician and the philosopher; unfortunately the more fundamental the equation and the more drastic the modifications to it, the less likely is this insight to be possible, because it depends so much on having a wide range of phenomenology associated with this equation and its variants. Failing this experimental back-up, the instincts of the physicist become less secure, and even where it exists (for example the non-relativistic limit of quantum theory) one might have to introduce meaningless and discontinuous limits to effect the transition, and one may no longer understand what is happening at the mathematical level, or how the interpretation of the formalism must change as a result. For example the invariant Δ function of the relativistic theory is the commutator of the fields and that it vanishes for spacelike arguements is an expression of causality. It is singular on the light cone and that is perfectly acceptable. But in the non-relativistic limit we do not know what is supposed to happen because in this limit the two light cones collapse to a single surface and that is a spacelike surface. One might think that the non-relativistic fields should not commute at spacelike separation and that non-relativistic theory is intrinsically non-local but that is not quite what happens; the fields do commute at distinct points at equal times.

All of this is to argue that the vague feeling that mathematical physics should be formal and concrete and that the program of making it rigorous and exploring the mathematical structures which underpin it is basically a mathematical one of interest only to mathematicians, has no clear philosophical foundation; it has a foundation in tradition, traditions of both research and dissemination, and undoubtedly "playing around with equations" in the style of Dirac will remain a powerful heuristic engine in research. Vague and figurative mathematical theories are surely less likely to be false, because less precise, and

more likely to appeal to the practising physicist, because more flexible; but however justifiable in the context of discovery, if we are interested in the truth or clarity of a physical theory, one must also be interested in consistency and rigour. And as indicated above, I would like to suggest that there is even a question of objectivity, that even if classical logic is actually an anthropocentric thing and that mathematics is subjective in this sense, figurative mathematics is still more contingent on historical and psychological accidents.

If these arguments appear weak and inconclusive there remains a more straightforward motivation to pursue the abstract method, which is that the formalism of quantum mechanics is *ad hoc* and unmotivated. One just has to accept its various elements, a complete normed complex vector space, a ray correspondence with states, observables which are self-adjoint operators, the CCR's and Schrödinger equation, one just becomes familiar with the mathematical technology and eventually it seems natural and even inviolate. In fact if one looks at the history of the abstract approach one sees that it was in reaction to this that such methods were developed and gradually there developed the insight that one could actually derive these things from more intuitive although also more abstract assumptions about the world. Historically these assumptions were grounded in operationalism but it is an open question as to whether this is the best philosophical platform for them; I shall try to argue that one can do as well on a weakly realist basis (that is realism with conventionalist undertones) if one is prepared to tolerate a high level of abstraction in one's metaphysics, and it is for that reason that I shall use the term abstract to describe this approach. Clifford Hooker has instead made a distinction between those abstract approaches which do or do not make use of operational interpretations in his editing (Hooker [1975], [1979a,b]) of the source papers in (mainly) non-relativistic applications (logico-operationalist and logico-algebraic respectively). But there are few publications in the field, and even fewer in relativistic

physics, which do not make some reference to operational ideas (this is particularly true of the algebraic approach to quantum theory)[3]. Because of this and the limitation of Hooker's selection to the non-relativistic case I do not adopt this terminology, although I am in sympathy with the distinction that he draws.

[3] It appears to me that Hooker's claim, that "though by no means universal, philosophical thinking in the mainstream logico-algebraic tradition is dominated by non-operationalist thought, much of it explicitly realist" (Hooker [1979b]), is untrue. He cites Putman on logical realism and Bub on conditional probabilities (in the context of lattice theory) - neither of them mainstream in the algebraic tradition.

2.1 Overview

> And we extend our concept...as if spinning a thread we twist fibre upon fibre. And the strength of the thread does not reside in the fact that some fibre runs through its whole length, but in the overlapping of many fibres.
>
> L. Witgenstein

2.1.1 Origins of the algebraic theory

We recall (1.1.1) that Heisenberg's initial breakthrough of 1925 was heuristically and mathematically based on algebraic considerations; so too was most of Dirac's work in 1925-6, a period of astonishing creative output. There is no question that physicists were able to conceive of the external world as described by abstractly defined algebraic systems.

With the advent of wave mechanics the mathematics was predominately analytic and the physical intuitions classical. The development of the theory was henceforth fuelled by two things: first, to effect a correspondence with the algebraic methods of the matrix mechanics, and second, to bring together analysis and differential geometry. The one leads to functional analysis, and the other to the theory of covariant differential equations (and ultimately group representations on fibre bundles, as we shall see). Physically these mathematical developments were associated with the probabilistic interpretation of the theory on the one hand (the transformation theory), and the search for a relativistic theory on the other.

The two stimuli were diametrically opposed, and of the two the latter prevailed. Thereafter the probabilistic interpretation of the relativistic theory was only salvaged by the passage to field theory.

Jordan was the only physicist to retain interest in the abstract methods which had proved so useful to the developement of the matrix mechanics. In (1.4.6) we briefly described the background to his return to the idea of an abstract algebraic system. The attempt was remarkably successful; he created *sui generis* an entirely new branch of algebra, and lay the foundations for a powerful new approach to quantum theory. The dominant *physical* input in this creation was a consideration of the limitations of measurement. It is a truimph of the operationalist philosophy that this input should have played such a pivotal heuristic rôle.

Just how fundamental and powerful a framework he had developed was not to be apparent for another thirty years; impressive results in the finite-dimensional case were soon to follow, however.

In his first paper on the new algebra Jordan proceded from assumptions on the possibilities of the organization and processing of laboratory data, to conclude that whilst the operations of addition and power-raising are operationally well-defined[1] (and should correspondingly be defined for the "observables" that occur in the formalism of the theory), this is not true of the operation of multiplication for non-commuting observables. The operation defined by $\frac{1}{2}((A+B)^2-A^2-B^2) = AoB$ is the nearest analogue of multiplication that can be defined: one thus obtains a *commutative non-associative* algebra of observables.

The rest of this paper (Jordan [1932]), and the two that followed ([1933a, 1933b], explored some mathematical aspects of such algebras and points of contact with the matrix

[1] Are they? Can one really measure the sum of two non-commuting observables? Can one even measure an arbitrary observable? These problems have not been resolved after decades of controversy. For a discussion of experimental tests of such propositions in the context of hidden-variable theories, see Redhead [1981].

mechanics (for example with the algebra of Pauli spin matrices). The conceptual weakness of these innovations resided in Jordan's exclusive reliance on the notion of an observable; one measures c-numbers in the laboratory, and to make transparent a relationship between the functional calculus (and hence algebraic structure) appropriate to measured c-numbers (which respects the limitations of simultaneous measurement of non-commuting observables) with the algebra of observables one needs to be able to pass from q-numbers to c-numbers - one needs in short the notion of *state.* At times Jordan indicated that the relationship was defined via the expectation values of observables, but this interpretation was not systematically developed. His most specific comments concerned the significance of the so-called *associator* $\{A,B,C\} = (AoB)oC - Ao(BoC)$ and the proof that $\{A,B,A^2\} = 0$; this relationship is

> that *part* of an associative algebra valid, and which has an unambiguous physical meaning, as a relationship between the various expectation values of the given quantum mechanical system. (Jordan [1933b p.288]).

Immediately following he also claimed:

> That the statistics of all measured magnitudes of a quantum mechanical system is determined only by quasimultiplication (i.e. the product defined above) - without complete knowledge (Kenntnis) of the multiplication - can also be understood as: *the interaction of a system with a macroscopic system only depends on the quasimultiplication.*

The implication is that somehow the usual non-commutative multiplication "discovered" by Heisenberg operates on the quantum level but only the quasimultiplication influences the results of measurements.

Jordan was not ready to jettison the non-commutative Heisenberg matrix algebra at this stage. It would have seemed perverse to do so, given the central importance of Heisenberg's contribution to the development of the matrix mechanics (and it is uncomfortable to recall that his "discovery" of the (associative, non-commutative) matrix algebra was also supposed to be guided by operational considerations).

As we have seen ((1.4.6)) von Neumann had himself produced a rudimentary axiomatization of quantum mechanics along these lines in order to formulate his "impossibility proof" of hidden variables; in his collaboration with Jordan and his friend and associate Eugene Wigner these interpretational questions were postponed and the mathematical structure of r-number algebras (as they then called them) explored for its own sake. In this spirit they began from an abstract algebra with quasimultiplication only, and then considered the circumstances in which it may be imbedded in an algebra of Heisenberg type (in which case let us call the algebra *special*). So we have an abstract algebra $\mathcal{U}$ and the assumptions (Jordan *et al* [1934]):

1. $\mathcal{U}$ is a real linear vector space.
2. for each $A \in \mathcal{U}$ and positive integer n we can form A^n so that the usual rules for polynomials are valid. In particular $\mathcal{U}$ then has a commutative distributive product o.

It was also assumed that $\mathcal{U}$ is "formally real", that is:

3. If $A^2 + B^2 + \ldots = 0$ then $A = B = \ldots = 0$.

From this simple axiomatization, which, (apart from the assumption of distributivity), was claimed to necessarily arise from the physical conditions of laboratory procedures, they were able to prove the remarkable result that for such algebras with finite dimensional basis the imbedding is always possible with a single exception: the algebra $\mathfrak{R}_3^8$ of all hermitian matrices of order 3 over the quasi-quaternions (Cayley numbers.) They were also able to show that *every* special real Jordan algebra of finite dimension is isomorphic to the real numbers, or hermitian matrices with one of the reals, the complex numbers, or the quaternions as entries. Subject to this ambiguity one has obtained essentially one of the classical or Heisenberg algebras as the *unique* representations of the r-number algebra.

Following a heuristic discussion of Jordan [1932] they were also able to obtain a spectral theory. The fundamental theorem is as follows.

Theorem 2.1.1

Let $\mathcal{U}$ be an r-number algebra. Then there exists in $\mathcal{U}$ a unit element $\mathbb{I}$ such that $\mathbb{I}oA = A$ for all $A\epsilon\mathcal{U}$. Furthermore each element A in $\mathcal{U}$ can be written in the form $A = \Sigma a_i P_i$, where $P_i^2 = P_i$, $P_i o P_j = 0$ for $i \neq j$; $\Sigma_i P_i = \mathbb{I}$, and a_i are the "proper values" of A, defined as the μ distinct simple real roots of the characteristic equation $f(a) = 0$ where $f(A) = A^{\mu} + \lambda_{\mu-1}A^{\mu-1} + \lambda_{\mu-2}A^{\upsilon-2} + \ldots + \lambda_1 A + \lambda_0$ is the polynomial of the lowest degree in A which vanishes.

This theorem points to a close connection between the algebra of observables and the set of idempotents (projections) in the algebra. The question must have arisen: what is the structure of this set? We shall take up this question in a moment.

Von Neumann returned to the abstract algebraic approach two years later in his paper "On an algebraic generalization of the quantum mechanical formalism Part 1" [1936][2]. This work was also related to the mathematical results obtained in his collaboration with Murray on rings of operators and published in the same year. The most obvious limitation of the r-number theory was that the results and much worse the methods of the r-number theory were limited to algebras with a *finite* basis, so they obviously couldn't apply directly to quantum mechanics. He considered that it was essential to introduce some topological structure in order to handle the infinite dimensional case, and in this paper introduced a weak operator topology (defined purely algebraically) for that purpose. However he was not able to make comparable progress towards a representation theory, and it was thirty years later that a classification for what are now called JW-algebras was developed (Stormer [1966]). The direction that von Neumann was pursuing at this time seems to have turned resolutely away from the concept of state. Since our elaboration of the algebraic approach *will* employ this concept (and our review is of necessity selective) we

[2]Part 2 was never published; nor was it found in his files.

shall not discuss this line of approach further[3].

2.1.2. Origins of the lattice theory.

The algebraic approach thus begun in so promising a manner lay fallow for more than a decade. It awaited in particular the further development of the representation theory of Banach algebras, a rapidly developing field following von Neumann's collaboration with Murray. Prominent in this area were George Mackey at Harvard and Irving Segal at Chicago.

Segal made important contributions to the representation theory of C^*-algebras and pioneered its applications in QFT; he also created an axiom scheme that (almost) leads naturally to a C^*-algebra. Mackey's achievement was in some ways even more impressive. From the mathematical point of view he not only made important contributions to the algebraic representation theory but he also extended the representation theory of abstract groups to the non-compact non-Abelian case (thereby completing the representation theory for abstract groups begun by Frobenius to include *all* the classical groups)[4]. His direct contributions to physics were made in an axiomatization of quantum theory[5]; this played a pivitol rôle in the subsequent development of the so-called *quantum logic* approach.

Although the Mackey and the Segal theory are very similar from a realist point of view (compare (3.2.2) with (3.3.5), Mackey's emphasis on the *probabilistic* theory of the *idempotents* of the algebra $\mathcal{U}$ lent the development of the

[3] For a fascinating review of this material and particular von Neumann's ideas concerning the so-called "continuous geometries" see Holdsworth and Holland [1983].

[4] From the point of view of foundations in physics the importance of the Mackey imprimitivity theory as applied to the spacetime groups cannot be over-emphasised; it is reviewed in Section 2.4.

[5] According to Jammer [1974 p.384], he was motivated to do so by his contact with Segal.

theory a fundamentally different character[6]. We have returned to the question of the structure of the set of idempotents in the algebra; like so much else, it was von Neumann who first took up this question, and as in the case of the r-number theory, gave a complete solution in the finite -dimensional case. If one considers the r-number algebra it is obvious that the idempotents are not a sub-algebra; of course it is the linear space structure which is not preserved (the sum of two idempotents is not an idempotent in general). The important step was to see that what remained was (in the finite dimensional case) essentially a *geometry*. This step was made by von Neumann in 1936 in collaboration with the American algebraist and lattice-theorist Garrett Birkhoff. They were to find that geometric methods could define the structure of the projection operators on a Hilbert space (completely, in the finite-dimensional case) and themselves provided a new abstract approach to quantum theory. The crucial point is that projection operators are in one-one correspondence with the closed sub-spaces of a Hilbert space, so that their structure is given by the structure of these subspaces - which is to say the *projective geometry* of Hilbert space[7].

From our point of view as important was that they considered the corresponding geometric structure in the classical case, which is to say, on classical phase space. It must have been apparent from the r-number representation theory that if quantum mechanics (or a generalization thereof) was to arise

[6]There are two aspects to this; the one concerns the priority of the notion of property or proposition over that of an observable (which, setting operationalism to one side, can only be a rather subtle metaphysical question). The other concerns the use of a (generalized) notion of probability. The latter is the fundamental idea that distinguishes the algebraic from the lattic theory, for it leads to the postulate of countable additivity. Compare (3.2.2) with (3.3.4).

[7]Wigner's [1931] theorem on the (anti)unitarity of linear transformations which leave transition amplitudes invariant already indicates the fundamental character of the projective geometry of Hilbert space. Nevertheless this does not appear to have motivated von Neumann's interest in lattice theory, which developed as indicated above.

from the hermitian matrix representations then the real number representation should describe classical mechanics[8]. In the paper of Birkhoff and von Neumann [1936] we see the first attempt to explore the parallel abstract foundations of quantum and classical mechanics.

From these two steps came a radical new idea; that quantum theory is the natural expression of a *new, and non Aristotelian logic.*

Strangely enough, this idea must have naturally presented itself. The abstract theory of projective geometry was by then well-developed as a branch of *lattice* theory; but this theory was constructed from its inception as a theory of logic. In particular it had long been known that the propositional calculus has the algebraic structure of a *distributive* lattice. Von Neumann and most partiuclarily Garret Birkhoff must have been familiar with this fact. They also knew, from the same mathematical background, that this structure is that of a field of sets (that is why Venn diagrams work). And the same intuition applies to the interpretation of a classical phase space (see in this connection (2.2.1)). The idea that in some sense a different *logic* is associated with quantum theory was no great leap of the imagination.

Nevertheless this idea does not fit very happily with von Neumann's general philosophy; in fact it was only in the concluding section (and a short one at that) entitiled "Relation to Pure Logic" that we see any hint of the logical realism that has subsequently been associated with the lattice approach. The predominant emphasis of the paper is operationalist, in the sense that lattice elements were interpreted as *yes-no experiments* or *experimental propositions*; but one also has reference to the partial ordering induced by set-theoretic inclusion as *logical*

[8] There remains the exceptional algebra $\mathfrak{K}_3^8$; the possible physical significance of this representation is still unknown.

implication and the comparison of the lattice to a propositional logic. At a crucial point the authors appeal also to a realist interpretation; that in the usual notation (definitions are given in the next section) although the meet and join of two *physical qualities* yields a *physical quality* a partial ordering between two such lattice elements is *not* a physical quality (but a relationship between physical qualities). This does flagrant violation to the propositional logic interpretation[9]; as a fact of lattice theory it has ever since been a thorn in the side of logical realists (and more generally of those who seek a logicist interpretation of quantum theory).

So these are the basic ideas of the 1936 paper: a subset E of phase space corresponds to the proposition that the system is located in E, and the classical Boolean algebra of the power set of a set corresponds to the Lindenbaum-Tarski algebra of the propositional calculus. The natural set-inclusion and set complement give this algebra the structure of a distributive complemented lattice. If then the analogous concept to phase space is Hilbert space and to subsets of phase space is subspaces of Hilbert space, this structure must be read off from the lattice structure of Hilbert space (i.e. its projective geometry). With partial ordering and the meet operations given set-theoretically, the join operation by set-union *and completion* (with respect to the linear space structure) to obtain a *closed* linear subspace, and with "negation" given by the orthogonal complement (as follows from de Moivre's laws), one obtains a *non-distributive* complemented lattice.

It is in this fundamental difference (bearing in mind the representation theory of Boolean algebras as a field of sets (Stone [1936] - a recent result known to the authors)) that one can seek to understand the intuitively peculiar features of quantum mechanics.

[9] That is, in predicate calculus, the sentances include all well-formed-formulas constructed from atomic predicates and the logical connectives, including material implication.

The simple example proposed by Birkhoff and von Neumann was as follows: using standard notation (definitions are given below and more formally in the next section), if a is the proposition that the state of an electron is on one side of a screen, $a^{\perp}$ that it is on the other, and b that this state is symmetric about the screen, then since $a \vee a^{\perp} = \mathbb{I}$ it follows that $b \wedge (a \vee a^{\perp}) = b \wedge \mathbb{I} = b$ whereas $(b \wedge a) \vee (b \wedge a^{\perp}) = b \wedge a = b \wedge a^{\perp} = \mathbb{O}$.

I shall not discuss the cogency of this example[10] nor of those which have followed it. But I would like to remark that the early expectation that apparent quantum paradox would simply disolve in the face of a careful abstention from the use of distributivity in natural language interpretations of quantum phenomena has proved too optimistic. Indeed, in the discussions of the idea that in quantum theory one needs a non-classical logic prior to the Birkhoff-von Neumann paper, it was always the *law of bivalence* (that one of two contradictory propositions is always true) that was rejected[11]; one feels that there has remained this tension within the quantum logic approach, that whereas it is bivalence that one most naturally wants to give up when confronted with quantum paradox it is distributivity which must actually fail.

2.1.3. Lattice representation theory.

From our point of view of greater importance to the abstract approach is that we now have[12] a second representation theory of an abstract system which once again yields a vector space as its unique finite-dimensional model. The sketch of the proof used the fundamental theorem

[10] Popper [1968] was later to make a number of criticisms.

[11] See the references of (1.4.6), fn. 49.

[12] The proof was actually incomplete; von Neumann and Birkhoff deferred the details to a later paper, which never appeared.

of projective geometry: that if $\mathcal{L}$ is a geometry[13] of dimension $n \geq 3$ then there exists a division ring D such that $\mathcal{L}$ is isomorphic to L(V,D), where V is a vector space of dimension n over D and L(V,D) is the lattice of subspaces of V. The important step was then to show that an inner product can always be constructed such that orthogonality (with respect to this inner product) is equivalent to the complement on the abstract lattice. This they were able to do: there must exist an adjunction on V (an involutary anti-automorphism) and a map on $V \times V$ which is sesquilinear relative to this adjunction such that the complement on the lattice is given by orthogonality with respect to this inner-product. This shows that such a lattice is isomorphic to the lattice of subspaces of a (finite dimensional) Hilbert space over the division ring D.

Concerning this division ring, there is once again the same ambiguity as in the r-number theory; D is one of the reals, the complex numbers, or the quaternions. We now have a partial explanation of this result: it is a theorem due to Kolmogorov that any projective geometry whose k-dimensional elements have a locally compact topology relative to which the lattice elements are continuous, must be over the reals, the complex numbers, or the quaternion field.

The idea that mechanics describes the logical structure of its appropriate phenomenology, and the idea that the content of quantum mechanics may be abstractly defined in such a simple and elegant language as that of lattice theory, have exerted a powerful fascination over the intervening half-century; the literature is now enormous. I shall not discuss the develop ments related to the first of these ideas; as goes the second, the *principal* develop-ments are twofold: first, in exploring the relationships between lattice theory and probability theory (Mackey, Gleason, Varadarajan). Second, in the discovery of a representation

[13] A geometry is an irreducible complemented modular lattice of finite rank; see Varadarajan [1968] for a beautiful introduction to projective geometry and its relationships with quantum mechanics.

theory valid in the infinite dimensional case which gives Hilbert space as the unique model for a certain type of lattice, and the attempt to motivate the assumption of this lattice structure on operational grounds (Jauch, Piron, Gudder).

The significant contributors came from a rather tight community: Varadarajan attended Mackey's lectures on these themes at the university of Washington in Seattle shortly prior to his first writings in the field, and Jauch expressly acknowledged the influence of his discussions and correspondence with Mackey. Jauch's influence in turn is enormous; amongst his students we find G. Emch, M. Guenin, J. Marchand, B. Misra, and C. Piron. But Mackey himself was influenced primarily by Segal. Although he was based at Harvard, he had extensive contact with Segal as a visiting professor at Chicago in 1955 from which dates his own investigations into the conceptual foundations of mechanics, to eventually find expression in his book "The mathematical foundations of quantum mechanics" published in 1963. Segal himself was directly influenced by von Neumann; his fundamental paper on the irreducible representations of operator algebras (Segal [1947a]) was submitted the year after he became assistant to Veblan at Princeton (where he had personal contact with von Neumann); this paper was closely related to the theory of operator algebras developed by von Neumann and was immediately followed by his definitive axiomatization of quantum theory (Segal [1947b]).

2.1.4. The Segal algebra.

Definitive, that is, within the algebraic approach, to which we now return. We recall that von Neumann's unfinished generalization of r-number algebras to the infinite dimensional case proceeded by the explicit introduction (at the axiomatic level) of a topological structure - a topology which was defined in algebraic terms alone. Segal's paper of 1947, entitled "Postulates for General Quantum Mechanics", used the same strategy but with a norm

topology instead of the weak topology introduced by von Neumann. The difference was crucial, not only from the point of view of obtaining a representation theory but also in the simplicity of his assumptions.

There are two kinds of postulates; the first are purely algebraic (in fact 1 and 2 above, with the additional assumption that $\mathcal{U}$ contains a unit $\mathbb{1}$). The second are topological: they make of the linear space introduced by the algebraic postulates a normed space and demand natural continuity properties of the algebraic operations with respect to this norm topology. The result is a real Banach space; if one then assumes an adjunction on this space one has a C^*-algebra and the GNS construction (after Gel'fand, Naimark and Segal) is then applicable to yield a representation theory for this algebra as a sub-algebra of $B(\mathcal{H})$, the set of all bounded operators on a Hilbert space $\mathcal{H}$ (for each representation the Hilbert space $\mathcal{H}$ is defined in a canonical way).

Here are Segal's postulates; we suppose that a dynamical physical system has an abstract structure $\mathcal{U}$ defined as follows:

The Segal Model

A. *Algebraic Postulates.*

1. $\mathcal{U}$ is a real linear space.

2. There exists in $\mathcal{U}$ an identity element $\mathbb{1}$ and for every $A \in \mathcal{U}$ and positive integer n an element A^n of $\mathcal{U}$, these being such that the usual rules for operating with polynomials in a single variable are valid.

B. *Metrical Postulates.*

There is defined for each observable A a non-negative real number $\|A\|$ such that:

1. If α is an arbitrary real number and A and B arbitrary elements of $\mathcal{U}$, then $\|\alpha A\| = \alpha \|A\|$, $\|A+B\| \leq \|A\| + \|B\|$. The vanishing of $\|A\|$ implies $A=0$; $\mathcal{U}$ is topologically complete when regarded as a metric space with the distance between A and B defined as $\|A-B\|$.

2. $\|A^2 - B^2\| \leq \mathrm{Max}\{\|A^2\|, \|B^2\|\}$.

3. $\|A^2\| = \|A\|^2$.

4. $\|\sum_{A\in\mathcal{R}} A^2\| \leq \|\sum_{A\in\mathcal{S}} A^2\|$ if $\mathcal{R}\subset\mathcal{S}$, $\mathcal{R}$ and $\mathcal{S}$ being finite subsets of $\mathcal{U}$ (this axiom has since been proved from the others).
5. A^2 is a continuous function of $\mathcal{U}$.

An abstract set $\mathcal{U}$ satisfying A1-2, B1-5 above is called a **Segal algebra.**

So much was explicit; but if one looks at the elaboration of the theory one finds that at every stage crucial use is made of the notion of *state.* States do not appear explicitly in his postulates (and we find, as in his later writings, the statement that "our theory is strictly operational in the sense that only the observables of the physical system are involved in the postulates." (Segal [1947b p.930])). States are actually introduced as a mathematically convenient tool for exploring the properties of the model, specifically as positive linear functionals on $\mathcal{U}$ (this way of introducing states was first proposed in von Neumann's book "The Foundations of Quantum Mechanics" [1932]). Remarkably Segal was able to prove that there *must* exist pure states on $\mathcal{U}$, and in fact there must be "enough" pure states to fully describe the algebraic properties of $\mathcal{U}$ in terms of the expectation values of observables in $\mathcal{U}$ (this is a gloss on the mathematically precise property that these states are *full* for $\mathcal{U}$. We shall see what this means in detail in Section 2.3).

The representation theorem is as follows: with $\mathcal{U}$ as above suppose an adjunction operation is defined (an involutary anti-automorphism) $A \longrightarrow A^*$. Suppose further that for all non-zero $A \in \mathcal{U}$, A^*A is positive definite; then $\mathcal{U}$ is isomorphic, algebraically and metrically, with a uniformly closed self-adjoint algebra of bounded operators on a real Hilbert space. If instead $\mathcal{U}$ is defined as above but with A1. replaced by: $\mathcal{U}$ is a *complex* linear space, then this is an isomorphism with an algebra of bounded operators on a *complex* Hilbert space. The Hilbert space $\mathcal{H}$ and the subalgebra of $B(\mathcal{H})$ are constructed from $\mathcal{U}$ itself when one is given any state on $\mathcal{U}$; therefore each representation is

associated with a particular state or class of states. This dependence of the GNS construction on states is something altogether new in the abstract approach; its proper interpretation was to await another two decades of research into the foundations of RQFT, and in certain applications (in particular to a theory of measurement) is still controversial. It is discussed in Sections 2.3, 3.5.

Segal himself thought the additional assumption of an adjunction "physically rather artificial", but still "less artificial " than the assumption that an observable just *is* a self-adjoint operator on a Hilbert space. From our point of view the difference is absolutely fundamental.

Segal did not articulate an operational motivation for these axioms; he referred to von Neumann [1932] in order to justify the algebraic axioms, whilst B1 was justified on the grounds of mathematical convenience: "If the system satisfied all the postulates except that of completeness, it could be completed in the usual way, and the resulting structure would then satisfy all the postulates, including that of completeness..." His other comments on the postulates were perfunctory, apart from the explicit definition of the norm in terms of certain (non-constructive) properties of $\mathcal{U}$. Explicitly, for any $A\in\mathcal{U}$ let $\phi(A)$ be the G.L.B. of the numbers α such that both $\alpha\mathbb{1}-A$ and $\alpha\mathbb{1}+A$ are squares. Then $\phi(A) = \|A\|$. In this sense he completes his "operational" interpretation: all the postulates express algebraic properties of an abstract set.

This operationalism I would like to call algebraism; the relationship between operationalism and the view that physical law should express relationships between observables only (in the technical sense of the term) is so weak that it cannot support this philosophy (for example: states are in one-one correspondence with idempotent observables in the usual axiomatization). What is actually at issue is whether one should permit the *explicit introduction of the notion of state at the axiomatic level.* Expressed in any other way the arguments become hopelessly

semantic: the fundamental operationalist assumption is that the primitive terms in the abstract theory (observables) are associated with laboratory operations. But how associated? Are they not associated with the *statistics* of experimental outcomes? It is not credible to argue from the basis of what is actually observed in the laboratory to a general philosophy which denies any fundamental rôle to the notion of probability (let alone the more general concept of state). On the most charitable view this can only be defended on the basis of parsimony; at worst, it is an altogether more mystical focus on the notion of pure algebraic structure.

To return to the Segal theory, the notion of state defined as a positive functional on $\mathcal{U}$ differs from that of the usual von Neumann axiomatization: for example there are dispersion-free states (or "generalized eigenstates") for observables with continuous spectrum. Segal claimed this was an advantage of his formulation of quantum theory, but it may have been viewed differently by his contemporaries. In any case, the theory was not widely appreciated outside of its applications in QFT. These ideas did, however, have a considerable influence on the Harvard mathematician George Mackey. Like Segal, the latter was led to an axiomatization of quantum theory having already developed powerful and original methods in pure mathematics, but in the representation theory of topological groups rather than normed algebras. Unlike Segal, Mackey did not exploit these representation theorems, although his axiomatization is essentially a generalization of classical probability theory (in the sense of Kolmogorov - that is as a measure theory), and uses similar techniques to the imprimitivity theorem.

2.1.5. The Mackey theory.

From our point of view Mackey's work is of interest because of its conceptual similarities to the algebraic theory that we develop in Section 2.3, whilst at the same time it highlights the principal mathematical differences between algebraic and lattice approaches. The latter is tied to

conventional Hilbert space quantum mechanics; we therefore have an abstract characterization of the differences between conventional theory and the C^*- algebra theory. For these reasons the following section is devoted to the Mackey theory and its connection with lattice theory. This will also provide an opportunity to develop the realist interpretation that we advocate in the possibly more familiar context of lattice theory.

Here we take a first look at the Mackey theory. In the original formulation (Mackey [1963c]) the axioms were as follows: let $\mathcal{B}$ be the set of all Borel subsets of the real line $\mathbb{R}$. Suppose we are given two abstract sets $\mathcal{O}$ and $\mathcal{G}$ and a function p which assigns a real number $p(A,f,E)$ in $0\leq x\leq 1$ to each triple A,f,E where $A \in \mathcal{O}$, $f \in \mathcal{G}$, $E \subseteq \mathcal{B}$. Physically $\mathcal{O}$ is to be thought of as the set of all observables and $\mathcal{G}$ as the set of all states. $p(A,f,E)$ is the probability that a measurement in the state f of A will yield a value in E.

The Mackey model

1. $p(A,f,\varnothing) = 0$, $p(A,f,\mathbb{R}) = 1$, $p(A,f,E_1 \cup \ldots \cup E_j \cup \ldots) = \sum_{j=1}^{\infty} p(A,f,E_j)$ whenever the E_j are pairwise disjoint.

2. If $p(A,f,E) = p(B,f,E)$ for all f and E then A=B; if $p(A,f,E) = p(A,g,E)$ for all A and E then f=g.

3. For any A and any real-valued Borel function u on $\mathbb{R}$ there exists $B\in\mathcal{O}$ such that $p(B,\alpha,E) = p(A,f,u^{-1}(E))$ for all f and E.

4. If $f_1,f_2,\ldots$ are in $\mathcal{G}$ and $t_1+t_2+\ldots = 1$ where $0 \leq t_i \leq 1$ then there exists f such that $p(A,f,E) = \sum_{j=1}^{\infty} t_j p(A,f_j,E)$ for all E and A.

The next two axioms focus on the idempotent observables in $\mathcal{O}$. Let $P \subseteq \mathcal{B}$ be the set $\{0,1\}$; then A is such if $p(A,f,P) = 1$ for all f. Now by 3 we might as well write B as u(A); let χ_E be the characteristic function of E; then $\chi_E(A)$ is the idempotent (or **question** in Mackey's terminology) which has eigenvalue 1 whenever a measurement of A yields a value in E and 0 otherwise (this is reminiscent of the operational interpretation of projections used in the theory of von

Neumann and Birkhoff; Mackey also suggested the terminology **yes-no** experiment). Questions have a natural partial ordering $Q_1 < Q_2$ whenever $p(Q_1,f,P) \leq p(Q_2,f,P)$ for all f; it is now a familiar step to define the "negation" of a question Q by $\mathbb{I}$-Qand to say that $Q_1$and Q_2 are **disjoint** ($Q_1 \perp Q_2$) if and only if $Q_1 < \mathbb{I}-Q_2$ (meaning they cannot have simultaneous yes values).

Let $q: E \longrightarrow q_E \in \mathcal{O}$ be any function from $\mathcal{B}$ to $\mathcal{O}$ such that if $E \cap F = \emptyset$ then $q_E \perp q_F$, and such that if $E_i \cap F_j = \emptyset$ for $i \neq j$ then $q_{E_1} \cup q_{E_2} \cup \ldots = q_{E_1} + q_{E_2} + \ldots$, and $q_\emptyset = \mathbb{0}$, $q_{\mathbb{R}} = \mathbb{I}$; then q is a **projection-valued measure** on $\mathcal{B}$. We can now state the remaining axioms:

5. Let $Q_1, Q_2, \ldots$ be any sequence of pairwise disjoint questions; then $Q_1+Q_2+\ldots$ exists.
6. For each projection-valued measure q there is an A such that $q_E = \chi_E(A)$.
7. The partially ordered set of all questions in quantum mechanics is isomorphic to the partially ordered set of all closed sub-spaces of a separable, infinite dimensional complex Hilbert space.
8. For any non-zero question Q there exists a state f such that $p(Q,f,P) = 1$.

Some comments: (1) says that for each A and f $p(A,\alpha,.)$ is a countably additive probability measure on $\mathcal{B}$; (2) says there is no redundancy - that for every pair of observables there is a state which distinguishes them and for every pair of states there is an observable to which they assign distinct values. (3) defines a functional calculus for the observables (in parallel to the algebraic postulate A2 of Segal) and (4) demands that $\mathcal{O}$ contains impure states (convex combinations of pure states). (5) demands that for disjoint questions the sum is defined (this gives us a fragment of A1) but it also imposes a certain topological property on the space $\mathcal{O}$ (or viewed diferently a continuity requirement on the action of the states on $\mathcal{O}$); this postulate marks the parting of the ways between the algebraic and lattice approaches. (6) says that the questions are in one-one

correspondence to the set of all projection-valued measures (this axiom was subsequently shown to be redundant); the remaining two are obvious.

As Mackey remarks, the seventh axiom has a rather different character from the others (that it is in particular *ad hoc*); he conjectures that one might be able to derive it from the others or that at least one will obtain a number of possibilities which do not differ very much from it.

It was in this context - in particular in consideration of real and quaternionic Hilbert space formulations of quantum mechanics (that is, exploring the threefold ambiguity in the choice of division ring that we have already encountered) - that these ideas reached a slightly larger community, with an extensive series of papers in Helvetica Physical Activa and in the Journal of Mathermatical Physics[14]. It was also in this context that Piron first encountered the lattice approach; if the catalyst was the Mackey theory, Piron drew more on his background in projective geometry. His axiomatization essentially proceded from a somewhat deeper study of the geometry of the lattice of subspaces in Hilbert space (the recognition that this lattice is a weakly modular, or orthomodular, orthocomplemented irreducible atomic lattice; these terms are explained in the next section). He was also able to show that if in addition this lattice satisfies the so-called *covering law* then it is *isomorphic* to a Hilbert space (with the usual ambiguity over the division ring)[15]. This is simple to prove in the finite dimensional case (that is, for lattices of finite rank) but it is a remarkable achievement in the infinite case; no explicit toplogical assumptions are introduced at the level of the lattice (it is not clear how one would do this anyway) and yet one has this coordinatization with a Hilbert space, which in the infinite dimensional case can only be defined using topological assumptions.

[14] Finkelstein *et al* [1962, 1963], Stueckelberg *et al* [1960,1961a,b,1962].

[15] A result independently discovered by MacLaren [1964].

Another way of describing Piron's achievement is therefore that he gave an axiomatization of Hilbert space with no explicit topological assumptions. Of course the assumption that the lattice is *complete* (that the meet and join of arbitrary lattice elements exists) has a superficial resemblence to topological completion (on substituting the linear space structure of a pre-Hilbert space for the lattice structure; for a Hilbert space algebraic completion is the same as topological completion).

The subsequent decade saw repeated attempts to provide an operational justification for the defining properties of the sort of lattice which Piron had shown was isomorphic to a Hilbert space (and for the lattice operations and partial ordering). Of these the most influential writings are undoubtedly those of Jauch [1968], [1971], Jauch and Piron [1969], in which the (by this time) standard operational interpretation of lattice elements as yes-no experiments was elaborated to apply only to *type one ideal measurements* (in Pauli's terminology). On this assumption some motivation was provided for the assumption of the covering law.

2.1.6. On the relationship between the lattice and algebraic approach.

I shall not elaborate on the various arguments proposed to motivate the Piron axioms; in the view taken here the idea that one should be able to construct a unique Hilbert space realization from laboratory procedures and their logical inter-relationships is too confining (these things are necessary - perhaps - but not sufficient for the development of physical theory).

But if now we clear our minds of the philosophical polemic over the rôle of measurements, and also of the metaphysical debate between properties and observables, we see that the question of the relationship between the lattice and algebraic approach has significant physical implications, and that there are no compelling philosophical grounds on

which the one is distinguished over the other. There is no contradiction between them, so one can easily adopt a symthesis of the two, but since basically the algebraic theory is a generalization of the lattice theory that amounts to a restriction of the former (so that it is a Σ^*-algebra, or a von Neumann algebra of type 1, or a W^* algebra, or possibly some other C^*-algebra with special properties[16]) And in this sort of restriction resides a great deal of physics (all of it in quantum field theory). From a realist point of view, that is how it should be; we do not seek an account of the world from necessary truths or postulates based on metaphysical convictions, we only seek to understand what those postulates might be, from which flow the mathematical formalisms that successfully describe the world. The *belief* that there might exist such a set of postulates is, however, a metaphysical conviction which stands at the heart of the absract approach. It is a further demand that these postulates make sense as statements *about the world*[17] (realism) or as statements *about laboratory procedures* (operationalism).

2.1.7. On group representations.

I have left until last the theory of group representations. The development of this theory has played a much more dominant rôle in mainstream physics and is, at least on the face of it, quite disassociated from any particular philosophical position, or indeed from the attempt to reconstruct the foundations of quantum mechanics. Nevertheless this theory stands at the heart of the abstract approach and perhaps explains or illuminates more hard physics than any other area of study. It is also from a

[16] For example Plymen [1968 a,b], Deliyannis [1972]; see also (2.3.6), (2.3.10).

[17] In (2.3.5) I shall formulate postulates which are in part statements about how the world can be described. To some extent these are contingent upon the nature of the world, but they also reflect choices that we make in how the contingent reality is to be described. To this extent the philosophy is weakly realist.

mathematical point of view closely tied to algebraic methods and the classification theory of Hilbert space representations of infinite locally compact non-compact groups (*e.g.* the spacetime groups) grew hand in hand with the comparable representation and classification theory of operator rings (*cf.* the applications of the Murray - von Neumann theory of factors and the "global" theory of group representations (Dixmier [1951], Kaplansky [1951], but most especially Mackey [1953]). Gel'fand and Raikov's study of the space of all irreducible representations of certain kinds of non-compact groups was essentially an application of the GNS method via an application of the Krein-Milman theorem to the convex set of all positive definite functions on the group (the space of states, in algebraic terminology). Most of the mathematicians involved worked in both fields, of course.

At this level, philosophical questions have little importance (*i.e.* with respect to the post-war development of the theory). In relationship to physics, the basic point is that from qroup theory alone we cannot conclude anything very definite about the type of representation, all we can do is classify the type of representation[18]; one has to stipulate the type. In quantum theory this is of course a (true or projective) representation of the group on a complex Hilbert space. Obviously the essential connection here is due to Wigner; but form a more general point of view, that is considering the origins of abstract methods, these geometric methods have in common the view that group theory is the study of representations, of groups which are abstractly given, a view that arose very slowly and more or less coincidently with the realization of the immense fertility of the idea of an abstract group[19]. Wigner's

[18] In the Galilean case that is really quite illuminating, one can see that there are no true representations of the group at all for positive mass, there are only projective representations; that is why there is no classical Galilean covariant scalar field theory with non-zero mass (cf. fn. 2 Section 1.4).

[19] The history of this developement therefore begins early in the nineteenth century. Partly for this reason, and partly

insight of [1931] was essentially to connect the abstract group with structural properties of Hilbert space, that is its lattice of subspaces. In this way, the abstract method subsumes the entire theory of group representations, for one can simply demand that the group has an action on the abstract algebra (or lattice) and the representation theory for the algebra (lattice) will fix the type of representation of the group.

How is the group to act on the abstract model? The most natural (and almost universal) assumption is that the group action is an automorphism of the model. This is equivalent to the assumption that the (algebraic or lattice theoretic) structure of the system is invariant under the group action and in particular for the spacetime groups this means that the structure of the system is invariant under dynamical evolution[20]. The implications of particle creation and annihilation phenomena are obvious: "the system" cannot be a single particle (and this reflects an aspect of the limitation of the lattice theory approach). On the other hand one might argue that the kinematic (free) theory enjoys a conceptually privileged status within dynamics and that it is appropriate that a foundational study of dynamics should be limited to the free theory (for example on operationalist grounds that our observations are limited to systems which are stable and in fact effectively free in an asymptotic sense). This view has an obvious relationship to the philosophy of the S-matrix approach[21].

because there exists an enormous historical literature on the group concept, I shall not attempt to trace its roots. A rudimentary outline is given in Section 3.4. Because the later history, in particular in relationship to algebraic representation theory, is of primarily mathematical interest, I shall not peruse it in any detail. Some further background is given in Section 2.4 and 3.1.

[20] As discussed in (2.1.2), this insight is due to Wigner [1931]. For a systematic and detailed exposition of the theory see Bargmann [1964].

[21] But as should be clear from the introduction, it is not for this reason that we restrict ourselves to the quasi-free and linear theories.

2.2 The Mackey Approach

> Our scientific information is summed up in measures. Science has at last revolted against attaching the exact knowledge contained in these measurements to a traditional picture-gallery of conceptions which convey no authentic information on the background and obtrude irrelevancies into the scheme of knowledge.
>
> A. Eddington

> In regard to Kant's philosophy, I beleive that every philosopher has his own Kant.... Arbitrary concepts are necessary in order to construct science, as to whether these concepts are given *a priori* or are arbitrary conventions, I can say nothing.
>
> A. Einstein

2.2.1 Motivation; realism and operationalism.

Before embarking on this realist reconstruction of the Mackey and Segal axiomatizations we should be absolutely clear about the motivation. It is this: we maintain it is a necessary consequence of the premiss that physical theory apply to *all* phenomena that one can apply dynamical theory to the measurement process itself. Even if we accept that what is to count as a "phenomenon" is that which can be empirically determined, there is no *a priori* constraint on the detail (or level of resolution) with which we can empirically monitor such paradigmatic measurement processes as the action of an ionization chamber, except that which follows from the theory itself.

Of course on the basis of what we know about quantum theory we can expect that *no* amount of ingenuity and sophistication

in the design of such monitoring facilities will ever permit us to assign precise numerical values to all observables. That is so because *with* such numerical assignments *assuming they can be regarded as a parameterization of the physical system in the standard way* (that is as Cauchy data for its subsequent evolution) one would have a hidden variable theory for quantum physics. The example of Kochen and Specker, and the Bell inequality, then give a more or less precise account of the sense in which this parameterization must be non-standard if one is to secure agreement with the quantum theory[1].

But, given this constraint on the simultaneous empirical determination of quantum observables, there can be no limit to the detail with which we can observe the complexities of measurement processes. The transition from the quantum to the classical levels of description may appear discontinuous, but surely the distinction is not recognized by the physical systems themselves; it is most particularly at the level of borderline phenomena which display both quantum and classical features that a wealth of phenomenology awaits discovery. In this exploration one must use physical theory at every stage; it is therefore essential that the interpretation of the theory *facilitates* this application.

This is just what the operational interpretation fails to do; it is philosophically incoherent to first *define* quantum observables as laboratory procedures and then to use the theory so-constructed to analyze these self-same procedures. It is even more incongruent that what appears from the outset as the most elementary and primitive mathematical objects (projection operators, for example) should then turn out to be enormously complicated mathematical models (of the

[1] There are a number of subtleties to this analysis; particularily as regards the relationship of the two - contextualism on the one hand, and non-locality on the other (Haywood and Redhead [1985]). We adopt the view that such non-standard parametrizations of quantum systems create more problems than they solve.

measurement process, analyzed in detail); that is what must happen if one both defines these primitive terms as measurement procedures and then permits the detailed analysis of *physically realistic* measurement processes.

To object at this point that there are two different things here - the *measurement process* as used to define the abstract primitives and the measuring process as it actually appears in the laboratory - is I maintain philosophically unsound. One might try to explicate the notion of *measurement process* in terms of an idealized measurement process, but I suggest that the evidence that *only systems of large numbers of degrees of freedom* can function as measurement apparati is conclusive. If this is recognized, the metaphysical nature of this idea of an idealized measurement process becomes obvious.

In any case this idea of an abstract measurement process is foreign to the positivist basis of operationalism. Once this is admitted, an operationalist philosophy can only appeal to measurement processes and it becomes pointless if not inconsistent to apply the theory to a detailed investigation of them. The alternative that we pursue is therefore to accord to the primitive entities of the abstract theory a *microphysical ontological* status. In now motivating the postulates of the abstract approach different criteria come into play. For example, it is in no sense our goal to establish that the postulates are *necessarily true.* On the other hand, if it is the ultimate empirical success of the whole theory that justifies the postulates of the theory, in a realist construal it is essential that the postulates can be understood as hypotheses about *the way the world is.*

But why has the operational approach proved so successful in furthering the abstract approach? I suggest that the reason is as follows: *if* we should be so lucky as to have a simple correspondence between primitive terms which refer to the microscopic level (observables) and certain properties of very carefully constructed macroscopic systems (measurement instruments) *then* this correspondence will impose a

functional calculus on the observables (as the image of the a priori calculus definable on the classical properties). It then seems reasonable to demand that this structure so determined should hold (on the microscopic level) *whether or not* the system is in interaction with a measuring apparatus; that is, I hold it is reasonable (and perhaps even essential from a realist point of view) to believe that in terms of dynamical law a measuring system is a physical system like any other. So if there is an operational constraint then it will also be a constraint on a realist interpretation (obviously the converse need not be true). One is left with a mystery: why should we be so lucky as to have a *simple* correspondence, even in special circumstances, between microphysical properties and macroscopic phenomena? I believe this must remain a mystery; it is equivalent to the question as to why science is possible at all.

One last point; if this is what realism amounts to (rather than something more specific, like subsumption under one of the two paradigm examples of classical physics, point particles or c-number fields on spacetime), then why has there ever been any conflict between quantum theory and realism? The answer is surely that one is uncomfortable at having to posit such a sophisticated and concrete an object as a *Hilbert space.* It will not do, and we know that it will not do (from the point of view of the probability interpretation of NRQM), to interpret this space as just a function space (with elements interpreted as c-number fields on spacetime). In the classical case the situation is different. Why is this so?

Hilbert space seems at first glance a sophisticated mathematical structure: an infinite dimensional normed vector space over a division ring which is complete in the norm topology, and where the norm arises from an inner product.

However, classical phase space is also a sophisticated mathematical structure. It is a C^{∞}-manifold in a metric topology, endowed with a Borel σ-algebra of sets generated

by this topology. In any application, even for the simplest systems, it is further assumed that there exists a symplectic structure on this manifold. Why, then, is the notion of a classical phase space considered natural and philosophically non-problematic whereas a Hilbert space is not?

The answer to this question is surely that the notion of a point in a 3-dimensional Euclidean metric space is a simple abstraction from macroscopic experience in accordance with our intuitions. One can simplify this intuition even further: to that of an element of a set, and step by step build up the structure of a classical phase space. The Borel algebra appears as a consequence of the intuition of the structure of the subsets of a set, and we may suppose the topology arises from spacetime intuitions (that spacetime is *continuous*), and thereby that it is a smoothly differentiable manifold.

It might be objected that this adds nothing to the fundamental intuition of a point in space. But how might one convey this intiution otherwise? You have a blackboard and a piece of chalk. A child of six asks you to explain the idea of a point in space. You may wave your hands, or you might draw a little picture of a box and mark a point in it. If the latter, you might move the point to different places, this movement would be continuous, and you might indicate that the point is contained (or can be contained) in different parts of the box. *A point in space.* As adults, the idea is simple; we should not underestimate the wealth of experience we have at our disposal. As so many concepts of natural language, its detailed explication may appear a torturous process.

If this is accepted, then one must also accept the conclusion that *if* it is possible to give a similar decomposition of the idea of a Hilbert space, such that its conceptual parts are statements about what the world is like, then the fact that we have no *other* way of expressing these intuitions should not count against their realist

contstrual; we have no subjective microscopic experience to which we can appeal. For this reason, we expect that "what the world is like" may be contrary to our macroscopic intuitions, and we will only be driven to *believe* that the world is indeed as described if the empirical evidence becomes overwhelming.

I believe that in this way one removes the central *intuitive* resistance to a realist construal of quantum theory (that is, given a "conceptual decomposition" of the concrete mathematical objects fundamental to quantum theory). It is in carrying out this task that as yet we can claim only partial success: there remain difficulties in measurement theory and in the interpretation of EPR-type correlations[2]. One can, however, address the problem of measurement in a coherent way and make some significant progress (we refer to the C^*-algebra techniques, as first exploited by Hepp [1972]; see (3.5.6)). The basic structure of quantum theory in the absence of measurements is, moreover, quite simply and intuitively defined. Further, one has as a result a new way of formulating the relationship between RQFT and NRQFT, and a general framework for the analysis of locality (the imprimitivity theory). These are studied in Section 3.

Having developed the broad philosophy let us now explore the most simple abstract approach to Hilbert space, on the basis of its geometric structure. The basic premis of the reconstruction proposed here (a variant of Mackey's [1963c] approach) is that in the microphysical domain one cannot insist a priori that the notion of a point in a set has any direct physical meaning, but that *assuming* there exists some kind of "event space" sufficiently structured to admit a notion of *state,* one can thereby define natural generalizations of all the classical intuitions connected with a classical phase space.

In the Mackey approach the basic idea is to develop the

[2]Nothing in this thesis will address the interpretation of EPR correlations.

theory as a generalized probability theory; that is, we assume that the states are *probability measures* (countably additive functionals normalized to unity). It is a *generalized* probability theory because we do not assume that this probabilism is epistemic in origin. For that reason it might be better to think of quantum probabilities as measures of "partial truth", "potentiality", or "latency"; we shall use the term *amplitude. A priori* it is not at all obvious how one might go about doing this. The trick is to take two characteristics of classical (Kolmogorov) probability theory and interpret them more abstractly. One is the fact that probability measures have a natural convex space structure which permits the distinction between *pure states* and *mixtures* (or *impure states*); in particular the pure states are extremals of the convex hull of all probability measures (*i.e.* cannot be written as the convex sum of any set of states). We interpret the pure states as those which carry *maximal information*; in particular classically they correspond to a "complete" specification (that is, in accordance with what we consider to be an exhaustive description of the world). In this way (by insisting that the states have a convex structure) we shall consider that the description of any system is "complete" just in so far as the state which fixes this description is pure.

The other key input is that the states are *countably additive* on the "event space" of the system, as in the classical case. This assumption is much harder to motivate. Additivity alone is, however, very natural; it expresses quite a fundamental intuition, which is that the "probability" of a set of mutually exclusive alternatives is the sum of the probability of each. Note that this means that we *can* apply the notion of probability to this *set.* It must make sense to talk about a "set of alternatives" (or as we shall say *properties*) as *itself* an alternative property. It is for this reason that *countable* addtitivity says something about "limit properties" (defined in terms of arbitrarily large sets of properties), but at the same time we have very little control over these limits, we do not

have a topology.

Nevertheless, if these assumptions are made (and we only postulate just enough to implement them) it turns out that with only a few supplementary assumptions or regularity conditions (and similar conditions are needed in the classical case), the resulting structure is isomorphic to either a Hilbert space or a classical phase space.

We do not have a categorical interpretation of this notion of generalized probability; of course it provides a way of parameterizing a physical system, just as the assignment of a collection of real numbers is a way of parameterizing a system. But beyond that, we suggest, it is only through experience with the theory (and particularly its phenomenology) that one can hope to develop an interpretation. This seems to be consistent with realism, and perhaps even a virtue to recommend the approach (consider the slow evolution of the concept of the *electromagnetic field*, from its initial grounding in Newtonian mechanics to an entity in its own right).

In particular, only at a later stage of the analysis should one try to tackle the question as to how this probability theory can ever come to approximate, in certain circumstances, the classical probability calculus, or more generally be interpreted in terms of limiting frequencies of observable phenomena.

2.2.2 The Mackey postulates.

Our primitive terms are **state, observable,** and **amplitude.** The physical theory of a system Ω consists of the parameterization of Ω with respect to the values of its observables. **A state is a specification of the amplitudes with which the observables of Ω take values in the Borel sets of the reals.** Let $\mathcal{G}$ be the set of states of Ω, and $\mathcal{U}$ be the set of observables. We introduce the notation $f(A,E)$ for the amplitude that $A \in \mathcal{U}$ has value in $E \in \mathcal{B}(\mathbb{R})$ when Ω has the description $f \in \mathcal{G}$. The first fundamental assumption is that for any A in $\mathcal{U}$, $f(A,.)$ as a map on $\mathcal{B}(\mathbb{R})$ *respects the Borel structure on* $\mathbb{R}$ (so it is in particular a *probability measure* on $\mathbb{R}$). This automatically gives us an algebraic structure on $\mathcal{U}$ since for any real Borel map $u(x)$ we can define $u(A)$ as that observable which has amplitude $f(A,u^{-1}(E))$ for each $f \in \mathcal{G}$ and $E \in \mathcal{B}(\mathbb{R})$ (so $u(A)$ has the same amplitude - whatever the state - for having values in E as does A in $u^{-1}(E)$).

This is the distinctive feature of the Mackey approach, that one "lifts" the algebraic structure of the real number line to the set $\mathcal{U}$ in this way. If one now proceeded to make $\mathcal{U}$ into a vector space (by defining a sum composition law for $\mathcal{U}$) we would be following the Segal algebraic approach. But we cannot do this and have a linear action of the states on $\mathcal{U}$ for fixed E, for obvious reasons (the amplitudes would no longer take values in the interval $[0,1]$). To get a linear space structure, we need a different way of describing Ω, and one loses the algebraic input from the algebraic structure of $\mathbb{R}$.

One does get a composition law for certain types of observables, however. To see this, we next introduce the idea of a **property** of Ω. If $\mathcal{G} \ni f$ is the description: "$\mathcal{U} \ni A$ has value in $E \subseteq \mathcal{B}(\mathbb{R})$ with amplitude $0 \leq \lambda \leq 1$" then we say that Ω has the property "A has value in E" with amplitude λ. By virtue of the action of states on $\mathcal{U} \times \mathcal{B}(\mathbb{R})$ we can define for each observable A a large class of properties, in one-one correspondence with the Borel sets E of $\mathbb{R}$, namely

the observables $\chi_E(A)$ (which we shall write more simply as A_E, meaning the property "A has value in E"); we can also pick out the properties in $\mathcal{U}$ by the requirement that A is a property whenever $f(A,\{0,1\}) = 1$ for *every* f (that is, A has value in the set $\{0,1\}$ in every state; one is immediately led to paradox if one thinks it then follows that A has value 1 or value 0 in every state).

The properties associated with each observable again inherit the Borel algebra of the reals in the following way: if E_1 and E_2 are disjoint then $f(A,E_1) + f(A,E_2) = f(A,E_1 \cup E_2)$; but then $f(A_{E_1}) + f(A_{E_2}) = f(A_{E_1 \cup E_2})$ and this is true for *every* f so that we can just define $A_{E_1 \cup E_2}$ as that element of $\mathcal{U}$ which has these amplitudes. We denote it $A_{E_1} + A_{E_2}$. It follows of course by construction that it is a property. This is the first step towards fixing the algebraic structure of the set of all properties, which we denote $\mathcal{L}$.

We tackle the problem of the *objectivity* of these descriptions in the following way: we suppose that it is possible, from any set of states, to form their linear convex sum (for the pair f_1, f_2 that is states of the form $\lambda_1 f_1 + \lambda_2 f_2$ where $\lambda_1 + \lambda_2 = 1$), and that in this way we obtain **statistical states** on $\mathcal{U}$. **We suppose that a description is subjective or may admit an ignorance interpretation only if it is given as a statistical state** (note that this condition is *not* sufficient). Therefore states which cannot be so written as a convex sum are **pure** or **maximal** (denote Θ^P).

Obviously to define a real generalization of classical probability theory we have to give up some part of the classical theory. The "law of bivalence" expresses the intuition that the set of all properties of Ω may be grouped into pairs and that a maximal description will specify that Ω has exactly one of each pair; one can trace almost all of the conceptual difficulties of quantum theory to the apparent violation of this intuition. We therefore consciously abandon it.

In its place we assume something weaker: if a property P has amplitude 1 (respectively 0) in some state f we shall sometimes say "P is true (false)"; consider now the property 1-P. It follows by construction that for any state the amplitude for P and the amplitude for 1-P sum to unity; 1-P has the natural interpretation as the *contradictory* property to P (negative, opposite or **complement**); we shall denote it $P^{\perp}$ ($\perp$ is in fact an **orthocomplement**).

We need two further postulates to complete our axiomatization; the first is a natural requirement of parsimony, that there is no redundancy in our descriptions (that for any two states at least one observable has distinct amplitudes) and that we have enough of them (that for any two observables at least one state assigns them distinct amplitudes). If this were not so, one could simply define equivalence classes of such states and observables in this way and then form the quotient spaces. In the same spirit we also require that Ω *can* have *every* property; that is, there are no properties in $\mathcal{L}$ which are *never* true in any state of Ω (the parallel assumption for states is obviously not necessary).

The second is much harder to justify; it concerns the algebraic structure of $\mathcal{L}$, and extends the sum composition law to certain sets of properties (where the elements of each set is not defined by a single observable). These sets are defined as follows: as we have seen, there are properties P and 1-P whose amplitudes sum to unity in *every* description. We say that P_1 and P_2 are **disjoint** (denote $P_1 \perp P_2$) when their amplitudes sum to a value less than unity in every description. If we assume that amplitude-assignment on properties is a linear map on the properties in $\mathcal{U}$ with respect to this composition law, it is obvious that this composition law can only be defined on pairwise disjoint properties. Making this assumption, $P_1 + P_2$ can be *defined* as that property whose amplitude is the sum of the amplitudes for P_1 and P_2 for *every* state in Θ. But it is not obvious, for any finite set of pairwise disjoint properties, that for every state the amplitude of each sums to a real number

which is less than unity; nevertheless we assume this also, and even demand that it is true for countable collections of pairwise disjoint properties. If we have bivalence then this assumption must be true (this is obvious if one considers only the pure states $\mathcal{G}^P$; each maximal description specifies *every* property as true or false and hence for any pairwise disjoint set if *one* property is true then all the others must be false). Therefore this postulate is a weaker version of this idea. It is also apparent that the properties of the form P_{E_1} for pairwise dijoint E_1 constitute such a class (and in this case the existence of $P_{E_1} + P_{E_2} + \ldots$ does not have to be postulated). In the quantum logic interpretation, this composition law is interpreted as **disjunction**. We also have the interpretation that Ω has the property $P = P_1 + \ldots + P_j + \ldots$ if and only if Ω has exactly one of the properties P_j.

With this postulate we have the central feature of the Mackey approach, which is that with this algebraic structure on pairwise disjoint properties we have just enough to make the states in $\mathcal{G}$ σ-additive normalized measures on $\mathcal{L}$; we have a generalized probability theory on $\mathcal{L}$. We have also obtained this theory in a way such that for certain subsets of $\mathcal{L}$ (properties all of which are associated with a single observable and pairwise disjoint properties) this algebraic structure is just lifted from the Borel algebras of the reals, that is, the natural Boolean algebra of subsets of a set. These are just the sorts of projections which have a "classical" structure amongst themselves (there is no problem in thinking about quantum systems so long as one thinks only of properties which are all functions of a single observable or all disjoint from one another).

Although we based the construction of this algebra on the notion of disjointness we could have based it on the idea that $\mathcal{L}$ forms a partially ordered set (or poset), with partial ordering $<$ where for $a, b \in \mathcal{L}$ $a<b$ holds if $f(a) \leq f(b)$ for all $f \in \mathcal{G}$. It is clear that $a<b$ if and only if a and $b^\perp$ are disjoint (in which case b and a are also disjoint). Since this ordering (like the algebra on pairwise disjoint sets) is intrinsic to $\mathcal{L}$, in the sense that it is

independent of state, one might try and interpret it in terms of $\mathfrak{L}$ alone.

This is the viewpoint of quantum logic: the partial order expresses *logical implication*. Certinly for properties of the form $\chi_{E_1}(A)$, $\chi_{E_2}(A)$ with $E_1 \subseteq E_2$ the partial order has the character of a logical truth. We prefer to regard it as *part of the* physics *of the system* Ω (and we shall speak of this order relation as **physical implication**) The picture is like this; if we are given an observable of a system then we obtain a whole set of properties automatically, and there are logical relationships among them (of the form if A has value in E_1 then it has value in E_2 if $E_1 \subseteq E_2$). But if we are given a second observable we have no way of knowing what ordering exists between properties associated with A and those of B; that depends on the way A and B are defined, and *that* depends on a host of other factors (typically as generators of a group of transformations on $\mathfrak{U}$). Once we are given the two observables in some way (for example, explicitly, in some representation) then of course we can work out this ordering and this will be done in a logical way; but there may be nothing intuitively very obvious about the ordering that we end up with. It is a question perhaps of how developed one's physical intuition is; that is, of how well one understands the physics of the system, mathematically and phenomenologically.

In summary the partial ordering can express necessary relationships between properties (in particular those associated with a single observable) or contingent ones (but independent of the state of the system). In both cases the order relationship certainly has the interpretation: if $a<b$ then whenever a is true of Ω then b is true of Ω, independent of initial conditions. The importance of the partial ordering derives from the fact that (for suitable $\mathfrak{L}$) we can use it to extend the composition law + to *all* the properties in $\mathfrak{L}$. If this is possible then $\mathfrak{L}$ becomes a lattice. But even if we do not take this step, $\mathfrak{L}$ is already a very interesting object from a lattice theory point of view; it is in particular a σ-orthocomplete, orthomodular,

orthocomplemented poset.

First, to make sure that we have not cheated, let us look again at the Mackey axioms (having dropped Axiom 6 and 7):

1. $f(A,.)$ is a probability measure on $\mathcal{B}(\mathbb{R})$; that is, $f(A,\varnothing) = 0$, $f(A,\mathbb{R}) = 1$, $f(A,\cup_i E_i) = \Sigma_i f(A,E_i)$ for E_i pairwise disjoint.
2. Parsimony: if $f(A,E) = f(B,E)$ for all E,f then $A=B$. If $f(A,E) = g(A,E)$ for all A,E then $f=g$.
3. Functional calculus on $\mathcal{U}$: for any real Borel function u on $\mathbb{R}$ there exists an element (denote $u(A)$) in $\mathcal{U}$ for each $A \in \mathcal{U}$ such that $f(u(A),E) = f(A,u^{-1}(E))$.
4. To define statistical states (and hence pure states as extremals): for $f_i \in \mathcal{G}$ and given real numbers $\lambda_i \in [0,1]$ with $\Sigma_i \lambda_i = 1$ then there exists $f \in \mathcal{G}$ such that $f(A,E) = \Sigma_i f_i(A,E)$ for all A,E.
5. Algebra of pairwise disjoint properties: if $\mathcal{D} \subseteq \mathcal{L}$ is pairwise disjoint then there exists $c \in \mathcal{L}$ such that $f(c) = \Sigma_{a\in\mathcal{D}} f(a)$ for every $f \in \mathcal{G}$.
6. Ω *can* have any property; that is, for any $a\in\mathcal{L}$ there is at least one state f in $\mathcal{G}$ such that $f(a) = 1$.

2.2.3. Connection with lattice theory.

To begin with, recall some basic definitions: given an arbitrary poset $\mathcal{L}$ define the **join** $a\vee b$ (disjunction) of two elements a, b as the least upper bound of a and b; that is, for any $c>b$, $c>a$, then $c>a\vee b$. Define the **meet** $a\wedge b$ (conjuction) as the greatest lower bound of a and b; that is, for any $c<a$, $c<b$ then $c<a\wedge b$. If the meet and join of any two elements is an element of $\mathcal{L}$ then $\mathcal{L}$ is a **lattice.** If the meet and join of aribitrary subsets of $\mathcal{L}$ exists in $\mathcal{L}$ then the lattice is **complete**. If this is true for countable subsets of $\mathcal{L}$ the lattice is **σ-complete.** If this is true for any finite (countable) pairwise disjoint sequence in $\mathcal{L}$ then $\mathcal{L}$ is **orthocomplete** (**σ-orthocomplete**). We say that $\mathcal{L}$ is **generated** by a set S if the lattice operations applied to the set S yield $\mathcal{L}$.

An **orthocomplement** $^{\perp}$ on $\mathcal{L}$ is an involutive ($(a^{\perp})^{\perp} = a$), dual ($a < b \Leftrightarrow b^{\perp} < a^{\perp}$) automorphism on $\mathcal{L}$ which satisfies $a \wedge a^{\perp} = 0$, $a \vee a^{\perp} = 1$. If for any $a, b \in \mathcal{L}$ with $a<b$ there exists $c \in \mathcal{L}$, $c \perp a$, such that $b = a \vee c$ then $\mathcal{L}$ is **orthomodular.** If for any $a \in \mathcal{L}$ there exists b in $\mathcal{L}$ satisfying: for any nonzero c such that $c<a$ then $b<c$, then $\mathcal{L}$ is called **atomic** (and b is then an **atom**). We say that a **covers** b if $a > b$ and for any c with $a>c>b$ then either $c=a$ or $c=b$. $\mathcal{L}$ has the **covering property** if for every a and every atom p such that $a \wedge p = 0$, the element $a \vee p$ covers a.

$\mathcal{L}$ must have all of these properties, together with one more which we shall come on to shortly (irreducibility), if we are to establish an isomorphism with a Hilbert space. Can they be understood within the Mackey approach? To begin with, let us establish the result claimed above; it is obvious that $\mathcal{L}$ is a poset, and that it is orthocomplemented. The sum operation for pairwise disjoint properties is in fact a join; the property c postulated by axiom 5 is the least upper bound of all the properties in X [3]. This means that $\mathcal{L}$ is orthocomplete.

To see that $\mathcal{L}$ is orthomodular, we note that if $a<b$ then a is disjoint from $b^{\perp}$ so there exists c such that $c = a+b^{\perp}$; but then $1-c^{\perp} = a+b^{\perp} \Rightarrow 1-b^{\perp} = a+c^{\perp} \Rightarrow b = a+c^{\perp}$. Clearly $c^{\perp}$ is disjoint from a.

Despite its technical appearance, the orthomodular property has a central rôle in lattice theory (and all lattice theory approaches to quantum theory). As we shall see, it expresses the idea that *a pair of disjoint properties can be described classically.*

This insight became possible following the various

[3] Proof: if for disjoint a,b, $a \vee b$ exists then $(a \vee b)^{\perp}$ is disjoint from a and b; by (5) there exists $c = a+b$ and $d = a+b+(a \vee b)^{\perp}$. But since $a<a+b$ and $b<a+b$ and by assumption $a \vee b$ is a least upper bound, $a \vee b < a+b$ so that $d > a \vee b + (a \vee b)^{\perp}$. Since $^{\perp}$ is an orthocomplement $d=1$, hence $a+b + (a \vee b)^{\perp} = 1$ so that $(a+b)^{\perp} = (a \vee b)^{\perp}$.

representation theorems for a **distributive lattice** (that is, a lattice for which the join and meet operations are distributive over each other). As we have seen, it was an early insight of Birkhoff and von Neumann that the failure of distributivity is the key to understanding the difference between quantum and classical mechanics (to which their example did not do justice). It had been proved only two years previously that every *Boolean algebra* is isomorphic to a field of sets and conversely (Stone [1934]) (any distributive lattice is a Boolean algebra). But the concept has a more complex history than that. As a fact about the geometry of an *infinite* dimensional Hilbert space (and specifically as a result in functional analysis) it was known in the early 30's (Stone [1932a]); but its purely geometric expression has been much delayed[4]. On the other hand it is closely related to the **modular identity** familiar from projective geometry (and modular lattices are certainly the best understood, after Boolean algebras). This states that for $a<c$, that $av(b\wedge c) = (avb)\wedge c$ for all $b \in \mathcal{L}$. In a three dimensional geometry if a is a line contained in a plane c then for any line b the linear span of a with the intersection of b with the plane c equals the intersection of the linear span of a and b with the plane.

Note that if $a<c$ then $a=a\wedge c$ so $av(b\wedge c)=(a\wedge c)v(b\wedge c)$; similarily $(avb)\wedge c=(avb)\wedge(avc)$, so that if $\mathcal{L}$ is modular then $av(b\wedge c)=(avb)\wedge(avc)$ and $c\wedge(avb)=(c\wedge a)v(c\wedge b)$ - the distributive laws. Von Neumann and Birkhoff called attention to this identity for the meet and join of *finite* dimensional sub-spaces of Hilbert space; they interpreted it as a weakened form of distributivity. Orthomodularity is a further weakening of distributivity; we say that b,c are a **modular pair** if for any $a<c$ the modular identity holds ($\mathcal{L}$ is therefore modular if every pair is a modular pair). For an orthocomplemented lattice orthomodularity is equivalent to: if $a<b$ then a,b is a modular pair.

[4] It was first formulated abstractly in connection with the Murray - von Neumann classification of von Neumann algebras, which proceeded essentially from the analysis of the various projection lattices defined by their idempotents.

A modular pair generates a distributive lattice (apply the modular identity to the elements $a, b, a^{\perp}, b^{\perp}, \mathbb{O}, \mathbb{I}$); as we shall see, a distributive sub-lattice can be classically described. Actually orthomodularity implies that any ordered (or disjoint) pair generates a distributive lattice; it does not imply that this is so for a pairwise disjoint *set* of properties, unless $\mathcal{L}$ is complete (Varadarajan [1962]). Note also that these conclusions also follow from the assumption that $\mathcal{L}$ is modular; the importance of the concept of orthomodularity is that it has this fundamental physical content, but extends to the infinite case.

Essential to the lattice approach is the assumption that $\mathcal{L}$ is at least σ-complete; there are reasonable intuitive grounds for this assumption[5]. With it one can abstractly describe a great deal of the conceptual structure of quantum theory - but one *cannot* obtain the Piron representation theorem without the additional properties of atomicity, separability, and the covering law. Since we shall not use the Piron theorem I shall not pursue their realist interpretation here.

To continue we must make one regularity condition : that for all f, a, b, if $f(a)=f(b)=1$ then $f(a \vee b) = 1$ (a Mackey model for which this is true we call **regular**).

[5] I hve in mind the approach of Beltrametti & Cassinelli [1981 11.6], which uses the idea that (what I have called) physical implication is sufficent to provide maximal descriptions of Ω. That is, for every $f \in \mathcal{G}$ there exists a $\in \mathcal{L}$ such that all the properties which f ascribes to Ω are those physically implied by the property a. This ensures atomicity and σ-completeness. Separability and the covering law appear the most difficult assumptions to interpret realistically.

2.2.4. The classical case.

First consider the intuition that a pure state will describe any property of Ω as true or false. For a σ-complete regular Mackey model, it then follows that $\mathcal{L}$ is distributive. It is then a Boolean σ-algebra. It is a strengthening of the Stone representation theorem for complete Boolean algebras that the σ-complete case can always be represented as an algebra of point sets (Loomis [1947]). More precisely, we have an isomorphism of $\mathcal{L}$ with a σ-complete Boolean algebra of the subsets of a set X modulo a σ-ideal in the algebra $\mathcal{L}$. (The connection with measure theory is obvious - the σ-ideals are the sets of measure zero; we shall shortly use this theorem for Segal models).

The set X is given abstractly; its points are actually subsets of $\mathcal{L}$. Further assumptions are needed to arrive at the differentiable structure of X. We suppose that these arise from spacetime intuitions. Within a pure lattice approach, it is perhaps not clear just how one should do this. From the Mackey theory , however, one already has the concept of a Borel measure; the *natural* Borel structure of the reals is canonicallly associated with the usual topology on the reals (as the smallest Borel structure which contains all the open sets of $\mathbb{R}$) which also underlies the notion of spacetime continuity[6].

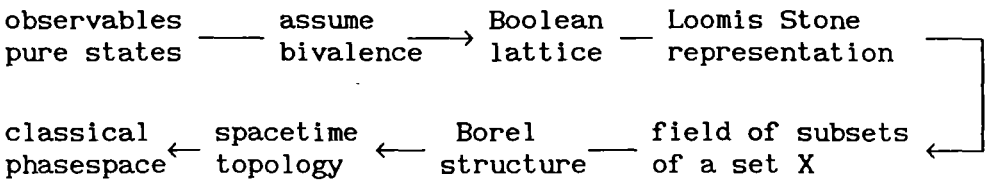
This point deserves brief comment. Given a topological group G and a left Haar measure μ then it is classical that for any Borel set $E \subseteq G$ with $\mu(E) \neq 0$, the set EE^{-1}contains a neighbourhood of G. The obvious implication is that the topology of G is completely determined by the Borel structure. It was Weil [1938] who first proved that this was so. But it was Mackey himself who proved the following sharpening of this theorem. Let G be a standard Borel group. If G admits a left (or right) invariant measure class, then

[6]This topology is, of course, locally compact and second countable. From now on by topology we mean a locally compact second countable topology.

G admits a left or right invariant σ-finite measure. Moreover, there exists exactly one topology for G which converts G into a topological group and whose natural Borel structure is the original one on G (Mackey [1957]).

We therefore see that topological assumptions which follow from spacetime intuitions are naturally introduced into the Mackey theory via the Borel structure that the algebra $\mathcal{U}$ inherits as we indicated earlier. This connection is particularly clear if we have a *position operator* which transforms in the intuitively obvious way under the Euclidean group. We shall study this theory (the imprimitivity theory) in Section 2.4.

Schematically we thus have:



It should be obvious that this entire circle of ideas goes through for a **distributive sub-lattice** of $\mathcal{L}$ (that is, a subset of $\mathcal{L}$ closed under the lattice operations). If such a sub-lattice is the *smallest* one which contains a countable set of properties $\mathcal{P}$ we say that it is **generated** by $\mathcal{P}$. If $\mathcal{P}$ generates a distributive sub-lattice then we say the elements of $\mathcal{P}$ are **compatible.** This is a symmetric, reflexive but *not* transitive relation on $\mathcal{L}$.

We now see in what sense orthomodularity is equivalent to the requirement that a pair of disjoint properties is essentially classical; it says that such a pair is compatible, and via the above representation theory we deduce their (simultaneous) classical description. The relationship to observables is as follows; the range of any observable A (as a map $\chi_E(A)$ of E into $\mathcal{L}$) is a Boolean σ-complete sub-lattice of $\mathcal{L}$, and compatible observables are those whose associated sub-lattices together generate a

Boolean lattice[7]. In this way Jauch, for example, in an axiomatization proceeding from the lattice alone, *defines* observables as "proposition-valued measures" (Jauch [1968])[8].

As for the states on $\mathcal{L}$, if $\mathcal{L}$ is a Boolean σ-algebra of subsets of a set X , and if $\mathcal{L}$ is **separable** (that is it is generated by a countable number of elements) *and contains the points of* X, one has a very natural result (Varadarajan [1968 Th.6.6]): For any point $p \in X$ let δ_p be the state in $\mathcal{G}$ defined by $\delta_p(a) = \begin{cases} 1 \text{ if } p \in a \\ 0 \text{ if } p \notin a \end{cases}$ for every $a \in \mathcal{L}$. Then the set of pure states $\mathcal{G}^p = \{\delta_p : p \in X\}$.

2.2.5. Connection with measurement theory

We end with a remark on the reduction theory of lattices. Define the **centre** of a lattice as the set $C \subseteq \mathcal{L}$ such that $a \in C$ implies a is compatible with every property in $\mathcal{L}$. The centre of a lattice is therefore the largest sub-lattice in $\mathcal{L}$ compatible with every subet of $\mathcal{L}$ (and if $\mathcal{L}$ is Boolean $C=\mathcal{L}$). At the other extreme C may be trivial (containing only $\mathbb{0}$ and $\mathbb{1}$); in that case $\mathcal{L}$ is **irreducible.**

If C is countable, one can effect a decomposition of the lattice $\mathcal{L}$ as follows; for each property a in C, consider all the properties in $\mathcal{L}$ which physically entail c (denote $\mathcal{L}_1$); likewise consider all the properties which physically entail $c^\perp$, $\mathcal{L}_2$ (for any $a \in C$ also $a^\perp \in C$). $\mathcal{L}$ can now be written as the direct sum of $\mathcal{L}_1$ and $\mathcal{L}_2$ (that is any $a \in \mathcal{L}$ can be written $a = a_1 \vee a_2$ with $a_i \in \mathcal{L}_i$). Proceeding in this way, one exhausts all the elements in C.

[7] The converse to this was Mackey's axiom 6 of (2.1.5); it was proved on the basis of the remaining axioms (excluding the Hilbert space axiom) by Varadarajan [1962].

[8] There is an even stronger result due to Varadarajan that compatible observables are essentially functions of one another (Varadarajan [1968 Th.6.6]).

In a heuristic way, one can see what happens if $\mathcal{L}$ is Boolean and atomic. In this case $C=\mathcal{L}$ and we can regard C as the set of all atoms in $\mathcal{L}$. The reduction sketched above will then give us a countable set of lattices (denote $\mathcal{L}(0,a)$) each one of which contains precisely 4 properties: $0, a, a^{\perp}, I$ (where a is an atom). Parametrize these atoms by the index $\lambda \in X$; the map $\tau:\mathcal{B}(X) \longrightarrow \mathcal{L}$ defined by $\tau(E) = \bigvee_{\lambda\in E} a_\lambda$ is then an isomorphism of $\mathcal{L}$ with $\mathcal{B}(X)$. The space X is the classical phase space of Ω; the points in X are in 1:1 correspondence with the elementary lattices $\mathcal{L}(0.a)$.

Let us make this notation more precise; for any $a,b\in \mathcal{L}$ we write $\mathcal{L}(a,b)$ as that lattice generated by the set $\{c\in \mathcal{L} : a<c<b\}$. If now C is a proper subset of $\mathcal{L}$ we will obtain a collection of irreducible lattices $\mathcal{L}(0,a_\lambda)$, $\lambda\in X$ as before but with a_λ no longer an atom (but note that each a_λ is an atom *with respect to the centre C* of $\mathcal{L}$: that is C, as a lattice itself, is atomic). These ideas are suggestive of the viewpoint that *each point in classical phase space is degenerate with respect to a set of associated properties which determine the microscopic structure consistent with this classical phase space point.* That is, the properties in C are the macroscopic classical properties associated with Ω, each one of which is associated with a class of "quantum" properties (with the relationships between them described by the lattice structure of this set). If this point of view is accepted (it will be elaborated later) one begins to see why the measurement problem has proved so difficult; if the time evolution is an automorphism on $\mathcal{L}$ this automorphism will also respect the *sub-lattice structure*, so that it will also be automorphic on each sub-lattice $\mathcal{L}(0,a_\lambda)$; therefore the evolution will necessarily leave the associated classical property a_λ invariant[9].

[9] The reduction of lattices can be very simply related to the superposition principle, which finds a natural expression within lattice theory. One learns that an irreducible lattice is one in which the superposition principle has unrestricted validity. The measurement problem, understood in terms of the transition from coherent superpositions to incoherent mixtures, is thus directly related to the reduction theory above. We shall study this transition in Section 3.5, but from the algebraic point of view.

2.3 The Algebraic Approach

> If I know an object I also know all its possible occurrences in states of affairs.
> (Every one of these possibilities must be part of the nature of the object.)
> A new possibility cannot be discovered later.
> Each thing is, as it were, in a space of possible states of affairs. This space I can imagine empty, but I cannot imagine the thing without the space.
>
> L. Witgenstein

2.3.1 Connections between ring and lattice.

As remarked above the crucial step in extending the algebraic approach was made in the development of a representation theory for Banach algebras (that is normed linear algebras which are complete in the norm topology). But even at this early stage it was recognized that knowledge of the lattice structure of a Banach space can provide a classification of such representations (one recalls that this idea was also central to the Murray-von Neumann classification of von Neumann algebras). It was soon appreciated that the *ring of all continuous linear transformations* on a vector space defines the representations of this space: if two normed linear spaces X_1 and X_2 are complete then they are isomorphic if and only if their associated rings $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are isomorphic (Eidelheit [1940]). The following year Mackey showed that one could drop the completeness condition in a similar result for the *group of automorphisms* on the space, and for the lattice of closed linear subspaces of the space (Mackey [1942]). As necessary conditions these theorems are trivial; to go the other way, the idea in both cases is to establish a correspondence between the abstract algebras (lattice or

ring) and the one dimensional sub-spaces of the linear spaces X_1, X_2, and then use the assumed isomorphism of the algebraic system to induce an isomorphism of X_1 and X_2.

A further fundamental result was obtained by Kakatuni and Mackey soon after: it is that X is isomorphic to a real (not necessarily separable or infinite dimensional) Hilbert space if one only knows that either the ring or the lattice admits an involutary anti-automorphism (or *adjunction)* which in the lattice (ring) corresponds to taking the orthogonal complement (adjoint operator) in the resulting Hilbert space (Kakatuni and Mackey [1944]). This theorem requires completeness; in the lattice case it extends the Birkhoff-von Neumann theorem for the finite dimensional case that we discussed earlier, with the obvious modification that the Hilbert space norm is equivalent to the original norm on the space X.

This theorem suggests a deep relationship between the adjoint operation and Hilbert space orthogonality, as also between the ring and lattice structures on X. First consider what is meant by an involutary anti-automorphism; on a lattice the fundamental structure is the partial ordering; an anti-automorphism *reverses* the partial ordering. Similarily in the case of a ring an anti-automorphism reverses the product operation. The additional assumption is that $a^{\perp} \wedge a = 0$ (for the lattice) and that $a^*a=0 \Rightarrow a=0$ (for the ring). A * **ring** is a ring R (*i.e.* a set endowed with two composition laws a+b and ab distributive over each other) together with the map *: $a \longrightarrow a^*$ satisfying $(a+b)^* = a^*+b^*$, $(ab)^* = b^*a^*$, $a^{**} = a$, with the condition $a^*a=0 \Rightarrow a=0$. An othocomplement on a poset is a map satisfying $a<b \Rightarrow b^{\perp} < a^{\perp}$ with the condition $a \wedge a^{\perp} = 0$. Therefore the existence of either a * operation on a ring or of an orthocomplement on a lattice is enough to force a (real) Hilbert space isomorphism.

In fact the ring of (continuous, linear) maps on a normed linear space X naturally defines a lattice which is isomorphic to the lattice of closed linear subspaces of X.

For any subset R of $\mathcal{R}$ let R^c be the set of **right annihilators** of R, $R^c = \{a \in \mathcal{R};\ ba=0 \text{ for } b \in R\}$. If for a given R there is a set $A \subseteq \mathcal{R}$ such that $R=A^c$ then we say that R is **closed.** It is a simple matter to show that the set of all closed subsets of $\mathcal{R}$ is a complete lattice when partially ordered by set inclusion. The * operation automatically defines an orthocomplement on this lattice $R = R^{*c}$ (Kakatuni and Mackey [1944 Th.3]; the equality is meant in a set-theoretic sense. In fact it also follows that this lattice is orthomodular). It is then obvious that the condition $aa^*=0 \Rightarrow a=0$ means that there can be no element a in both R and R^{*c}, so that their set-theoretic intersection must be zero, which is what the lattice condition amounts to.

If $\mathcal{R}$ is the set of all continuous maps on X it is straightforward to show that any closed set in $\mathcal{R}$ can be defined by a single element of $\mathcal{R}$ (as a^c) and that $(a^c)^{*c} = a$ when $a^*=a$ (*i.e.* when a is self-adjoint). If further a is idempotent ($aa=a$) then a^c is given by $\mathbb{I}-a$ (if $\mathcal{R}$ has a unit $\mathbb{I}$). Finally if $a^c \subseteq b^c$ then $b=ba$ so we can define the partial order independently of the underlying set theory for self-adjoint idempotents as $a<b$ if $ab=a$ (note that the relation $ab=a$ is reflexive, antisymmetric, and transitive).

In this way we can proceed more abstractly, if we forget about the definition of the * ring as the set of all continuous linear maps on X. For an arbitrary * ring *define* $\mathcal{L}$ as the set of self-adjoint idempotents and define an order relation as $a<b$ if and only if $ab=a$; with $a^\perp$ as $\mathbb{I}-a$ one can deduce $\mathcal{L}$ is an orthocomplemented orthomodular poset.

(which may, however, be trivial)

Any * ring gives such an $\mathcal{L}$; we have an immediate and very simple correspondance between *special* Segal models (*i.e.* those which can be imbedded in an associative algebra) with adjunction, and any lattice approach which postulates that $\mathcal{L}$ is orthocomplemented and orthomodular.

To deduce that $\mathcal{L}$ is complete, Kakatuni and Mackey used the fact that $\mathcal{R}$ was given as the ring of *all* (continuous,

linear) transformations on X (and it is only here that the fact that $\mathcal{R}$ is a ring is used). More recently it has been shown that one can dispense with this additional information about $\mathcal{R}$ and actually make do with an abstract *semigroup* (that is with a single associative composition law); more precisely we need to pass from the set of right annihilators of some subset $R \subseteq \mathcal{R}$ to an element in $\mathcal{R}$. This can be done by the simple abstract postulate to this effect: for each $a \in \mathcal{R}$ there exists a self-adjoint idempotent e such that $\{b \in \mathcal{R}: ab=0\} = e\mathcal{R}$ (*i.e.* the right annihilator of a equals the right ideal generated by e). If this is so $\mathcal{R}$ is called a **Baire * semigroup.** If further for every $R \subseteq \mathcal{R}$ there exists an e such that $R^c = e\mathcal{R}$ we have precisely what we need to carry through the previous construction of a complete lattice; a Baire * semigroup with this property is called complete.

The upshot of this discussion is that the ring of continuous linear transformations on a complete normed linear space, if a * ring, forces a Hilbert space isomorphism; also that if the lattice of subspaces admits an orthocomplement, the same conclusion follows. And that if one forgets about the underlying space X a * ring naturally defines an orthocomplemented orthomodular lattice (which may not be complete); and finally that one can even do this starting only from a * semigroup if only it is Baire (in the last two cases the lattice is complete $\Leftrightarrow$ the semigroup or ring is complete in the sense defined above)[1].

[1] There is an extensive literature on the use of Baire * semigroups in the axiomatization of quantum theory. The fundamental theorem here is due to Foulis [1960]: every orthocomplemented orthomodular lattice generates a Baer * semigroup which by the construction above defines a lattice isomorphic to the one we started from. But he also proved that we do not define a Baire * semigroup uniquely in this way, because starting from the semigroup, constructing the lattice, and then constructing the semigroup, we do not have this isomorphism. The lattice does not determine the semigroup completely. We also mention the operational expression of the postulates of Baire * semigroups due to Poole [1968], that a generalized expression of the Luders - von Neumann projection postulate on the conditioning of states by Pauli type 1 measurements (that is that one projects the state onto the subspace spanned by the properties that have been measured and renormalized to unity) exists and has many of the features of a Baire *

2.3.2. Digression: on infinite dimensional systems.

Mackey and Kakatuni had developed a representation theorem for a *real* Banach space X given the existence of an involutary antiautomorphism on either the ring or lattice structure of X. Some four years later they considered the analogous situation in the *complex* case. Unlike the reals (which has only the identity), the complex field possesses a great many discontinuous automorphisms, either linear or antilinear; but it has only *two* continuous ones, namely the identity or complex conjugation. Their result (Kakatuni and Mackey [1946]), that a transformation on X which carries closed subspaces of X into closed subspaces must be continuous as an automorphism on the complex number field, meant that the involutary anti-automorphism must be complex conjugation on the number field.

This result depends critically on X being *infinite* dimensional, unlike every result which we have discussed so far. Moreover Kakatuni and Mackey presented counter-examples in the finite case. The dimensionality of X - in particular that in physics it is infinite-dimensional - derives from the assumption that dynamical variables can assume a continuum of values (this seems intuitively more obvious than that variables can assume arbitrarily large numbers of values, which has the same consequence). From a kinematic point of view, one could equally say that it derives from the assumption that spacetime is continuous or infinite. From the abstract point of view, it is a significant success that one can *uniquely* define the operation of complex conjugation - concretely defined on $\mathbb{C}$- to a simple, abstractly characterized map on the fundamental algebraic structure of a physical system.

semigroup. In particular the multiplication of the semigroup is defined in terms of repeated conditioning, or more specifically, compositions of maps on the set of states $\mathcal{S}$ which have the action of the projection postulate.

2.3.3. Origins of the GNS construction.

We make the connection with the Segal theory if we observe that the ring of operators considered by Kakatuni and Mackey was actually a complete ring of operators in a certain topology on $\mathcal{R}$ itself (of course lifted from the topology on X, as the weakest topology such that the elements of $\mathcal{R}$ are continuous on X - the weak dual topology or weak * topology). The fundamental idea of the Gel'fand, Naimark and Segal theory is to suppose that $\mathcal{R}$ is given as a normed topological space from the outset. Gel'fand and Segal had actually been investigating the relationships between locally compact groups and certain self-adjoint operator algebras; in particular Gel'fand and Rykov [1943] had shown that such a group has a complete set of strongly continuous irreducible unitary representation (a set of representations is called **complete** if there is no element except zero which is mapped into zero by every representation). The self-adjoint algebra in question possessed a natural *linear space* structure in addition to being a ring, which was inherited from its definition in terms of *functionals* on the group (rather than linear transformations; that is , linear maps to the reals or complex numbers). The linear space structure of these number fields induces a linear space structure on the algebra.

This is exactly what happens in the Mackey theory, where the states are functionals on the observables of the system. It applies equally to the Kakatuni-Mackey theory, that of course $\mathcal{R}$ can be considered a vector space as well as a ring by defining $\lambda a(f) = a(\lambda f)$, $a \in \mathcal{R}$, $f \in X$, with λ in the division ring of X. (The fundamental difference between the GNS construction and the Kakatuni-Mackey theory is that one supposes the algebra to come equipped with a norm topology and not a weak topology; see also (2.3.6),(2.3.10)).

We shall shortly develop this representation theory. For the moment we should mention the fundamental feature of this theory, that the representations obtained are in correspondence with the states on the algebra. Each state

canonically defines a Hilbert space and a representation of the algebra as a set of operators on that space. This fact is the central insight of the algebraic approach to quantum theory. It plays a fundamental rôle in our understanding of quantum field theory.

2.3.4. The Segal theory.

From the foregoing we find (Segal [1947a]) that an abstract algebraic system $\mathcal{U}$, which is a * ring and a linear vector space, complete in a norm topology, actually comes equipped with a complete representation theory if we are given a "large enough" set of states on $\mathcal{U}$. We might even guess that the use of a norm topology rather than a weak topology will lead to a different, less straightforward relationship between the completion of the algebraic system and the completeness of the orthomodular lattice that we know will be defined by a * ring. Such an algebraic system is called a **C^*-algebra** (complex case) or **R^*-algebra** (real case).

We have already discussed the Segal axiomatization. This work (Segal [1947b]) was written only partly to exhibit the general Hilbert space isomorphism theory for a C^*-algebra. He was in fact able to make significant progress with the elaboration and interpretation of the more primitive algebraic structure of a Jordan algebra (with only the commutative non-associative product), under the assumption that it is complete in a norm topology. *For such a system is, in the associative case, an R^*-algebra.* This is not, incidentally, as restrictive as it seems, for a non-associative algebra will always contain associative sub-algebras. To obtain the representation theory in the general case one has to assume the Jordan algebra can be embedded in an associative algebra (*i.e.* that it is *special*) and that an adjunction exists on this algebra; Segal himself considered this step physically unmotivated.

We also recall that Segal made no use of state at the axiom level despite its crucial rôle in his theory (and not only in the representation theory). That a proper exploitation of

the notion of state along the lines of the Mackey theory leads to a natural development of a C^*-algebra was first shown by Gerard Emch, in his book "Algebraic Methods in Quantum Field Theory" [1972]. The following is a reformulated version of the Emch axiom scheme; our interptetation, however, is entirely different.

2.3.5. Realist construction of a C^*-algebra.

The basic philosophy of our approach is described in (2.2.1); we now effect a reconstruction of quantum theory from an algebraic standpoint, rather than on a geometric/probabilistic basis.

The fundamental objects are once again a set of observables $\mathcal{U}$, and a set of states $\mathcal{G}$; but instead of the action of the states that we had previously we shall speak of the **measure, value, or possessed value** of an observable in a given state: for any $a\in\mathcal{U}$, $f\in\mathcal{G}$ there is a real number f(a). For certain states (*i.e.* in certain configurations of Ω) we shall be able to think of these real numbers as the **expectation values** of certain observables, but only when we can prove that they may be interpreted in terms of the statistics of experimental outcomes. At the axiomatic stage we shall use the idea that they are possessed values to motivate the algebraic structure; when this interpretation is not possible we suppose the idea of a measure primitive, that is, as a parametrization of the algebra which describes the objective structure of the physical system Ω. We suppose that further physical interpretation of the measure of an observable (when it is not a possessed value) should follow the develop~ment of a measurement theory.

To fix our intuitions let me anticipate the ideas that follow from Section 3.5. If we consider a measurement process, Ω must include the measurement apparatus, a system of "sufficiently many" degrees of freedom, for which a cetain class of observables (**classical observables**) have measures which *always* admits a probabilistic (ignorance or epistemic) interpretaton, and which are just the mean values

of the classical observable. The measures of microphysical systems ("observed" systems, or in our approach microscopic sub-systems of Ω) are correlated to the measures of the classical observables. In this way they themselves become susceptible to a probabilistic interpretation.

At this stage we only make the following assumption; we suppose that $\mathcal{U}$ has a unit $\mathbb{I}$ such that for all f in $\mathcal{G}$, $f(\mathbb{I})=1$. We can look on this as a normalization condition on the states or as a definition of a unit; if the former, we justify this assumption on the grounds that we can always construct non-normalized states at a later stage if the physical interpretation requires them.

The fundamental difference between the Mackey theory and the Segal-Emch axiomatization is that we lose the input that a state and an observable together determine a measure over the Borel algebra of the reals. With it we also lose the technique for generating functions of the observables (determined as observables with values in the inverse functions of Borel sets of $\mathbb{R}$). The algebraic postulates are not an abstract generalization of classical measure theory (although they do yield a measure theory in the classical case, when the algebra is associative or the C^*-algebra in which it is embedded is commutative).

For all that, the real numbers play a crucial rôle in the construction; in particular, when we can think of them as possessed values of the observables. This idea is so useful that we build it into our postulates as follows: for every observable $a\epsilon\mathcal{U}$ there exists a set of states $\mathcal{G}_a$ (called the **dispersion-free states of a**) such that for each $f\epsilon\mathcal{G}_a$, f(a) is the possessed value of a in the state f. This warrants a functional calculus: a^n is that observable with possessed values equal to the n^{th} power of the possessed values of a. To do this (as also to introduce the linear space structure) we need as in the Mackey theory a postulate of parsimony, such that knowing the measures of all observables is enough to define a state, and knowing the measures provided by all the states is enough to define an observable. Because of our

interest in the dispersion free states of an observable $a \in \mathcal{U}$ or a set of observables $\mathcal{F} \subseteq \mathcal{U}$ (denote $\Theta_{\mathcal{F}}$) we want to know when we can define an observable by proper subsets of Θ; accordingly we say that a subset $\Im \subseteq \Theta$ is **full** with respect to a set of observables when the algebraic relationships between measure-assignments of states in $\Im$ on this set then determine the algebraic structure of this set. Equivalently, that these algebraic relationships then hold for *all* states in Θ (which is of course a full set of states by assumption).

We can express these ideas concisely if we assume that the algebraic structure is completely determined by the natural partial ordering $a>b \Leftrightarrow f(a) \geq f(b)$ for all $f \in \Theta$. By parsimony we demand $f(a) = f(b)$ for all $f \in \Theta \Leftrightarrow a = b$; a set of states $\Im \subseteq \Theta$ is **full** with respect to a set of observables $\mathcal{F} \subseteq \mathcal{U}$ if $f(a) \leq f(b)$ for all $f \in \Im \Rightarrow a<b$, $a, b \in \mathcal{F}$.

We are most particularly interested in the possibility that a dispersion-free set of states (for some set $\mathcal{F} \subseteq \mathcal{U}$) is full for some (possibly distinct) set $\mathcal{B} \subseteq \mathcal{U}$. We naturally assume that this is so when $\mathcal{B} \subseteq \mathcal{F}$ for a *sufficiently large* set of dispersion-free states. We obviously need something like a "maximal" set of dispersion-free states. We formulate this idea as follows: a set of states $\Im \subseteq \Theta$ is called **complete** if it is full for every observable for which all the states in $\Im$ are dispersion-free. In other words, to be complete, a set of states must determine the algebraic structure of *all* the observables which have possessed values when described by any state in this set. We then say that $\Im$ is **definite** (or value-definite) for this set of observables (or any proper subset; note that if $\Im$ is definite it is certainly complete). Is the set of *all* dispersion free states of any observable complete? *We make this a postulate.* If further given two observables $a, b \in \mathcal{U}$ the set of dispersion free states which they have in common is complete, we say that a and b are **compatible.**

It is obvious from this definition that a compatible set of observables has an algebraic structure identical to that of

the real numbers. The algebraic combinations of the possessed values define algebraic combinations of the observables. We can immediately obtain in this way a commutative distributive algebra. For each observable we have a functional calculus, that is for each integer n and observable a there must exist an observable a^n such that $f(a^n) = (f(a))^n$ for every f in $\mathfrak{G}_a$, and this is enough to define a^n uniquely. Incidentally, it is now obvious that the states in $\mathfrak{G}$ must be *positive* maps on $\mathcal{U}$, because if they are maps to the reals, $f(a^2)\geq 0$ for every $f\in\mathfrak{G}_a$ and hence for every f in $\mathfrak{G}$.

To define the vector space structure for observables which are not compatible we proceed as in in the Mackey theory: for any $a,b\in\mathcal{U}$ there must exist $a+b\in\mathcal{U}$ such that $f(a+b)=f(a)+f(b)$ for all f in $\mathfrak{G}$ (and it is obvious that $\mathfrak{G}_{a+b} \geq \mathfrak{G}_a \cap \mathfrak{G}_b$, equality only obtaining when a and b are compatible). We are not looking for an algebra on the set of idempotents; there is no constraint on the sum operation as applied in the Mackey theory at this stage. For the same reason we can immediately define scalar multiplication in the obvious way: λa is that observable such that $f(\lambda a)=\lambda f(a)$ for every f in $\mathfrak{G}_a$.

We also note that the states are now automatically linear on $\mathcal{U}$, so that in full from the idea that they are maps to the reals and from the idea that they define the algebraic structure of observables by the algebra of possessed values we conclude that they must be positive linear maps on $\mathcal{U}^2$.

[2] Primas has declared that the principle weakness of the algebraic approach is that states are assumed sui generis linear functionals, that this approach makes tacit assumption of a statistical ensemble interpretation (Primas [1981]). There is sense to the question: why should a state be linear? only if one has an algebraic structure independent of the notion of state. This is what happens in the Jauch-Piron theory where it is a genuine achievement of the approach that Gleason's theorem guarentees that states linear on pairwise disjoint properties must be linear in the general case.
In the present approach, the objection of Primas finds no purchase.

The foregoing gives us a commutative Jordan product $aob = \frac{1}{2}((a+b)^2-a^2-b^2)$; the question arises as to whether this product is distributive over + and scalar multiplication. According to our general methodology since it is distributive over a compatible set of observables (and no undue restriction is involved) we assume this is generally the case.

There remains the topological structure of $\mathcal{U}$. There is a natural candidate for a norm of an observable, namely the supremum of its possessed values (we could equally assume it is the supremum of its measures; the two are the same). In this way one has a natural norm topology, hence $\mathcal{U}$ is a *metric space.*

At this point the question: are we to consider only *finite* observables (*i.e.* bounded in norm), or are we to admit the possibility that an observable has infinite possessed values? This is a question to which an operational philosophy supplies the clear specification that we exclude the latter; if observables correspond to laboratory operations, and "obviously" no laboratory operation can ever measure an infinite quantity, every observable must have finite norm.

Despite its apparent plausibility, this argument does not stand scrutiny. Is the question of whether or not spacetime is infinite an empirical one? The answer is surely affirmative, subject to the usual equivocation on the inductive under-support of theoretical statements. Certainly physicists regard the question as empirically meaningful, at least in the context of General Relativity. And of course it will be laboratory experiments that will decide the issue. Neither can one say that a physical system Ω is not the sort of thing which includes spacetime (the predominance of kinematics, particularly in the relativistic case, is such that most of quantum theory is about systems defined entirely in terms of their spacetime properties).

On a realist basis one must admit the possibility that

observables have infinite norm; but one can, as a convention, attempt to construct a fundamental theory which deals only with bounded observables, because one can always construct an unbounded observables from bounded ones, *once* one has a representation theory, by appropriate restriction of the domain of definition. With this conventional philosophy we henceforward suppose that the elements of $\mathcal{U}$ are *bounded. We also assume that* $\mathcal{U}$ *is complete in norm.* This postulate is *partly* conventional; as Segal remarked ([1947b p.932]) it is largely a matter of convenience, and were it not complete, we could complete the algebra at a later stage. The situation is similar to that for the real numbers: completion allows us to treat infinite sequences which converge and are Cauchy as elements in the algebra. For all that, there is an important difference, namely that there are other natural topologies that we can define on $\mathcal{U}$ (although they are not in general metric topologies). The choice of topology also has physical significance (the question is: what sorts of limit points are we to define in $\mathcal{U}$, and with what utility in physical applications?) Of course, a metric topology is in intuitive terms very natural (and one can think of convergence in terms of the Cauchy property); it is not so clear that the limit points introduced by the postulate of orthocompleteness or σ-completeness are "approximated by" elements in the algebra (see also (3.3.6)).

We therefore proceed pragmatically; the "correct" choice of topology (with respect to which we complete $\mathcal{U}$) is to be decided by applications and phenomenology. The norm topology is, however, very natural in the present context; it has also an immediate pay-off: the states on $\mathcal{U}$ are *precisely* the normalized positive linear *continuous* functionals on $\mathcal{U}$. That is, *any* positive linear functional on $\mathcal{U}$ is automatically continuous in this topology.

We list our assumptions:

The Segal model

S1. For every physical system Ω there exists a set of states $\mathcal{G}$ and a set of observables $\mathcal{U}$ such that $\mathcal{G} \geq f: \mathcal{U} \longrightarrow \mathbb{R}$, containing the unit and zero, and $f(1) = 1$.

S2. $\mathcal{G}$ is full for $\mathcal{U}$.

S3. For each observable $a \in \mathcal{U}$ and real number λ there exists the observable $\lambda a \in \mathcal{U}$ such that $f(\lambda a)=\lambda f(a)$ for all $f \in \mathcal{G}$. For any pair $a, b \in \mathcal{U}$ there exists $a+b \in \mathcal{U}$ such that $f(a+b) = f(a)+f(b)$ for all $f \in \mathcal{G}$.

S4. For every $a \in \mathcal{U}$ $\mathcal{G}_a$ is complete.

S5. For every $a \in \mathcal{U}$ and integer n there exists $a^n \in \mathcal{U}$ such that $f(a^n)=(f(a))^n$ for all $f \in \mathcal{G}$.

S7. $\mathcal{U}$ is distributive.

S8. $\mathcal{U}$ is complete in the norm $\|a\| = \sup_{f \in \mathcal{G}} \{f(a)\}$.

It is a straightforward matter to verify that the structure of $\mathcal{U}$ defined above satisfies the postulates 1,2,3,4,5,6 (weakened),7, of Emch [1972]. Accordingly we may use his Th.7 [1972 p.54] to prove the metric postulates 2,3,4,5 of Segal [1947b] (this theorem uses the weakened form of Emch's postulate 6). Consequently $\mathcal{U}, \mathcal{G}$ satisfying S1-S7 above also satisfies all of Segal's postulates; $\mathcal{U}$ is therefore a Segal algebra. It is also a (distributive) r-number algebra (in fact $\mathcal{U}$ as defined by postulates S1-S6 already satisfies the postulates of the Jordan, von Neumann and Wigner theory).

We call $\mathcal{U}$, $\mathcal{G}$, satisfying S1-S8 a **Segal model**. We do not yet have a C^*-algebra. A Segal model is, we maintain, the natural framework for the description of a set of physical qualities, which correspond to the elements of $\mathcal{U}$, in terms of their measures (defined by the states on $\mathcal{U}$), so that the algebraic relationships between these measures are, in the dispersion-free case, mirrored in the algebraic relationships between the elements of $\mathcal{U}$. In this way we may consider the Segal model a natural weakening of the classical realist intuition, that physical qualities may be associated with real numbers and the algebraic relationships between these real numbers is mirrored in the algebraic relationships between these physical qualities. On this philosophy, it is plausible that if physical qualities may

be associated with *complex* numbers then a corresponding weakening of the classical realist scheme will lead directly to a complex version of the Segal model[3].

I shall not proceed in this way, however, but simply make the natural assumptions that follow if the Segal model is to be considered a real algebra imbedded in a complex algebra $\mathcal{A}$. We suppose that $\mathcal{A}$ is a complex algebra, in the sense that it is an abstract set which is a complex linear vector space with product distributive, associative, but not necessarily commutative, equipped with an adjunction *. We suppose $\mathcal{U}$ is embedded in $\mathcal{A}$, in precisely the same way that the reals are imbedded in the complex numbers. This motivates the assumptions that $\mathcal{U}$ is the set of elements in $\mathcal{A}$ such that $a^* = a$, $a \in \mathcal{A}$ and that if $a^*a = 0$ then $a = 0$. We also require that the Jordan product aob in $\mathcal{U}$ is given by 1/2(ab+ba) (where ab is the product in $\mathcal{A}$) when a,b are considered elements of $\mathcal{A}$ (*i.e.* that $\mathcal{U}$ is special). A Segal model satisfying these additional requirements we call a **fundamental model**. In full it must satisfy the postulates:

The fundamental model

F1. $\mathcal{U}, \mathcal{G}$ is a Segal model.
F2. $\mathcal{U}$ is special; it is embedded in a complex * algebra $\mathcal{A}$ such that $aob = \frac{1}{2}(ab+ba)$.
F3. $\mathcal{U} = \{a;\ a \in \mathcal{A},\ a^* = a\}$.
F4. If $a^*a = 0$ then $a = 0$.

We note that a model $\mathcal{U}, \mathcal{G}$ satisfying the axioms S1-S8 with the exception of S7 and also F2-F4 is automatically distributive (that is, S7 can be dropped). For $a \in \mathcal{A}$ we may

[3] Is this a radical step? Undoubtedly from a mathematical point of view; this programme would require considerable innovations (which is why I do not attempt it). Philosophically, one must face up to the question of the reality of complex numbers. In QFT one deals with classical fields which are intrinsically complex, where this complex structure relates directly to empirical phenomena (charge and charge conservation; see (Section 3.4)). There seems no good reason for excluding complex structures from our ontology.

write $a = a^+ + ia^-$ with $a^{\pm *} = a^{\pm}$ (*i.e.* $a^{\pm} \in \mathcal{U}$) and define for arbitrary $f \in \mathcal{G}$, $f(a) = f(a^+) + if(a^-)$. In this way one has a straightforward transport of structure from $\mathcal{U}$ to $\mathcal{A}$. $\mathcal{G}$ is full on $\mathcal{A}$ and $\mathcal{A}$ is complete in the norm $\|a\|^2 = \sup_{f \in \mathcal{G}} \{f(a^* a)\}$. It follows that $\mathcal{A}$ is an (abstract) C^*-algebra, and the self-adjoint elements of $\mathcal{A}$ are the elements of $\mathcal{U}$. Since a fundamental model automatically implies the existence of the model $\mathcal{A}, \mathcal{G}$, we shall on occasion refer to this too as a fundamental model.

There are a variety of ways in which one might hope to introduce $\mathcal{A}$ in a more natural way; by introducing an orthocomplement in $\mathcal{U}$ and exploiting the Kakatuni-Mackey theory, by finding some natural condition on $\mathcal{U}, \mathcal{G}$ so that it is automatically special (*i.e.* an infinite-dimensional analogue of the result of von Neumann *et al*), or by proceeding directly from a complex version of our construction of the Segal model. On an intuitive basis, however, the construction above appears entirely satisfactory.

2.3.6. Connection with lattice theory.

At this stage we also establish points of contact between the foregoing and the Mackey theory. It is obvious that for a compatible set of observables the algebra of possessed values is associative; also it follows from our definition of compatibility that a compatible set of observables can always be enlarged to a (compatible) *sub-algebra* of $\mathcal{U}$, that is, the algebra that it generates is also compatible. We conclude that a compatible sub-algebra of $\mathcal{U}$ is associative. We shall shortly see that a representation theory for the associative case (which does not require any additional postulates to those above) gives every such algebra a concrete realization as a classical theory. We therefore expect that *associativity* is the algebraic analogue of *distributivity* in the Mackey theory. If we temporarily make this assumption, namely that a sufficient condition for a set of observables to be compatible is that the algebra which they generate is associative, and defining properties

in $\mathcal{U}$ as idempotents in $\mathcal{U}$, we have that for any two properties a,b in $\mathcal{U}$ the following are equivalent[4]:

(i) a and b are compatible.

(ii) the subalgebra generated by a,b is associative.

(iii)the lattice generated by a,b is distributive.

(iv) The associator $\{a,c,b\} = 0$ for all $c\in\mathcal{U}$.

It is also a simple result (Emch [1972 Th.12 p.67]) that for two properties $a,b\in\mathcal{U}$ $a<b$ implies that a and b are compatible. The poset defined by the idempotents in $\mathcal{U}$ is therefore orthomodular. It is *not* however in general orthocomplete; that is, although one can prove that a sequence of pairwise disjoint properties is compatible, and that the completion in norm of the algebra generated by such a set is compatible, this completion does *not* require that countable sums of the form $\Sigma_i a_i$ exist in $\mathcal{U}$ (unless one shows that the limit $\Sigma_i f(a_i)$ exists and that the sequence $f(a_i)$ is Cauchy). This point is intimately related to the failure of Gleason's theorem for a Segal model, in parallel to the fact that one does *not* require the states to be countably additive on (pairwise disjoint) sequences of properties in $\mathcal{U}$.

This point cannot be emphasised too strongly: the *fundamental difference* between lattice and algebraic methods resides in the difference between the way these spaces are completed. In the lattice theory, without any appeal to the notion of state, completion can only be defined algebraically and all the burden is thrown on the interpretation of the lattice operations; if one uses the notion of state, one can define various completions (in terms of a variety of topologies that one can then define on $\mathcal{U}$), in one of two ways: first, by continuity requirements of the action of the states on $\mathcal{U}$, or more directly, from the intuitive notion of "approximation" (*i.e.* convergence; this is what we have done, it is particularly easy with a metric

[4] Emch made the assumptions (i) ⇒ (iv) and (ii) ⇒ (i) his structure axioms 6 and 8 respectively. The first of these implies $\mathcal{U}$ is distributive, which is all that is required to make $\mathcal{U}$ a Segal algebra.

topology). One *can* define a topology on $\mathcal{U}$ with respect to which the Segal model becomes a Mackey model (Plymen [1968a]) or a Piron model (Plymen [1968b]; for this one needs a weak (operator) topology on $\mathcal{U}$ (which can be defined in an abstract, but not very perspicuous, way). If $\mathcal{U}$ is first countable (this is not always true[5]) one just completes in this topology. More generally, one must choose a slightly smaller space (essentially complete only for sequences; see (3.3.10)). It should be noted that we did not make use of the *purely* algebraic correspondence between a * ring and a lattice discussed in (2.3.1); having the concept of state we can define the partial ordering on the idempotents in $\mathcal{U}$ in the usual way rather than first imbedding $\mathcal{U}$ in an associative * algebra (to get a * ring) and then using the relation ab=a. The two methods agree, of course (but for the reasons just discussed we do not get a complete lattice).

2.3.7. On the notion of state.

Although the set of states $\mathcal{G}$ started life as the set of all maps from $\mathcal{U}$ to the reals, normalized to unity, because we defined the algebraic structure of $\mathcal{U}$ in terms of these measures as also the norm we automatically find that $\mathcal{G}$ is the set of all normalized continuous positive linear maps from $\mathcal{U}$ to $\mathbb{R}$. We now define pure states in $\mathcal{G}$, denote $\mathcal{G}^P$.

We do this as in the Mackey theory, except that we do not need to posit that $\mathcal{G}$ is a convex space because given a sequence λ_i of positive real numbers whose sum is unity the map $\Sigma_i \lambda_i f_i$ is automatically a state if each f_i is a state. $f \in \mathcal{G}$ is then **pure** if it cannot be written as a convex sum. The sort of question that then presents itself is how big does a set of observables have to be before a state which is dispersion-free on this set must be pure; unfortunately we cannot answer this question in this generality, but it does

[5] A complete metric space is always first countable; so if $\mathcal{U}$ is a Segal system it is first countable.

follow that for any observable and for any of its possessed values there is a pure state which is dispersion-free for that observable and which assigns it that possessed value (Emch [1972 Th 11 p.64]).

It is also a fundamental result that $\mathfrak{S}^P$ is *full* for $\mathfrak{U}$ (Segal [1947a Th 2 p.938]). Therefore the pure states in $\mathfrak{S}$ already determine the entire structure of $\mathfrak{U}$. This situation is satisfactory; if we think of states as a way of parametrizing a physical model Ω, then it is natural that it should be possible to characterize Ω in terms of weighted averages over possessed values and measures, but one would hope that this technique plays no essential rôle in the postulates of the theory.

Incidentally, Segal's postulates made no reference to states. But he was able to prove an existence theorem to the effect that there *must* exist a full set of pure positive linear functionals on $\mathfrak{U}$ (hence a full set of pure states), that is, he showed that given any two observables $a, b \in \mathfrak{U}$ one can construct such a map f with $f(a) \neq f(b)$.

The fundamental fact of the Segal theory and the GNS representation theory is that not all states of $\mathfrak{S}$ can be represented as density matrices on a single irreducible Hilbert space, even in the most favorable case (when Ω is an elementary system, in the group-theoretic sense). Physically there are for example pure states which assign a real number to (bounded functions of) the position and momentum operators. If one thinks of pure states as the "wave functions" of Ω in the present approach there exist "eigenfunctions" of the position and momentum operators. From a realist point of view this is very satisfactory.

The failure of Gleason's theorem ramifies through the GNS construction to a really very profound and radically new idea; it is that one must deal with a number of different representations of one and the same physical system. Very often it is possible to single out an essentially unique representation on the basis of further physical assumptions (a mild continuity requirement is enough for systems of

finite number of degrees of freedom), but in the infin. case this choice is a much more difficult question. It i particularly problematic when a quantum field is described in a thermodynamic limit or defined on a non-trivial spacetime manifold. Of course one cannot evade the problem by renouncing the algebraic approach; on the contrary it is only the algebraic approach which gives any real leverage on these problems, which present themselves anyway. Since the thermodynamic limit defines a classical limit and is intimately related to the measurement problem, and since it is of interest to establish the generality of the particle interpretation of a quantum field, these ideas are explored at length in the following.

2.3.8. Representation theory and the GNS construction.

It is essential to at least heuristically understand the basic ideas of the GNS construction.

First, a Segal model already comes equipped with a representation theorem when the algebra is associative (it is then an R^*-algebra[6] with adjunction given by the identity; since a Segal algebra is anyway commutative every automorphism is an anti-automorphism also). Since any observable generates a one-dimensional associative Segal algebra we at once have a spectral theory. This is now the standard way to prove the spectral theorem for self-adjoint operators on a Hilbert space.

Theorem 2.3.1.

Every associative Segal algebra $\mathcal{U}$ is isomorphic (algebraically and metrically) with the algebra $\mathfrak{C}(X)$ of all real-valued, continuous functions g on a compact Haussdorf space X (with the obvious algebra of addition, scalar multiplication and powers of a function). The norm is then $\|g\| = \sup_{x\in X} g(x)$, and every state χ in $\mathcal{U}$ has the form $\chi(g)=$

[6] That is, satisfying all the properties of a C^*-algebra except that it is vector space over the reals rather than the complex numbers.

$\int_X g(x)d\mu_\chi(x)$ where μ_χ is a regular Borel measure on X such that $\mu_\chi(X)=1$. Conversely, every such measure generates in this way a state in $\mathfrak{S}$. When $\chi\in\mathfrak{S}$ is pure $\mu_\chi(x)=\Theta(p-x)$, where p is a point in X and Θ is the step function (alternatively $\chi(g) = g(p)$ or $d\mu(x) = \delta(x-p)dx$). (Proof; Segal [1947a]).

We see that in the associative case we indeed find that states are measures, in the technical sense, despite the fact that our postulates do not require σ-additivity (essentially by application of the Riesz-Markov theorem which asserts that every positive linear functional χ on $\mathfrak{C}_o(X)$ (continuous functions on X with compact support) there corresponds a unique regular Borel measure μ such that $\chi(g)$ is given as above). It is, incidentally, clear that every pure state is dispersion-free for *every* observable in $\mathcal{U}$.

When $\mathcal{U}$ is the one-dimensional subspace generated by a single observable a we see that X is the set of possessed values; it is usually denoted σ_a and is called the **spectrum** of a. We also recall that the spectrum of a bounded operator is always *compact*.

This representation theorem is an application of the GNS theory to the associative case. In the non-associative case, we must assume that $\mathcal{U}$ is a fundamental model[7]. Suppose then that $\mathcal{U}$ is embedded in a C^*-algebra $\mathcal{A}$. The GNS construction then leads to the following representation theorem; for each

[7] It is possible to assume that $\mathcal{U}$ is embedded in a real associative * algebra rather than a complex one, that it satisfies F1-F4 but with F2 modified to the real case. The GNS construction may then be applied without restriction to obtain a sub-algebra of $B(\mathcal{H})$ where $\mathcal{H}$ is a real Hilbert space. In this way we may exploit the results obtained in consideration of the three-fold ambiguity of the Piron representation theorem (2.1.5). In this context Stueckelberg et al [1960],[1961a,b],[1962] have shown that one may always introduce a complex structure on such a real linear space to obtain the conventional quantum mechanics. We shall do something essentially equivalent in Section 3.4 but I do not attempt a detailed comparison with this work. Mention should also be made of a result of Emch [1972], that no uncertainty relationships are possible for operators on a real Hilbert space.

state f in $\mathfrak{S}$ there exists a canonical representation π_f of $\mathcal{A}$ as a sub-algebra of the set $\mathbf{B}(\mathcal{H})$ of all bounded operators on a complex Hilbert space $\mathcal{H}$. $\mathcal{U}$ is isomorphic to the self-adjoint operators in this sub-algebra.

First, recall some definitions; a map π: $\mathcal{A} \longrightarrow \mathbf{B}(\mathcal{H})$ is a **representation** if $\pi(\lambda a+\mu b)=\lambda\pi(a)+\mu\pi(b)$, $\pi(ab)=\pi(a)\pi(b)$, and $\pi(a^*)=(\pi(a))^*$ (that is, it preserves the algebraic structure of $\mathcal{A}$, hence $\mathcal{U}$). It is **faithful** if its kernal is $\mathbb{0}$. A representation π is **cyclic** if there exists a vector $\Phi \in \mathcal{H}$ such that the linear manifold $\pi(\mathcal{A})\Phi$ is dense in $\mathcal{H}$ in the norm topology on $\mathcal{H}$. It is **irreducible** if the only subspace of $\mathcal{H}$ stable (*i.e.* invariant) under $\pi(\mathcal{A})$ is $\mathbb{0}$ and $\mathcal{H}$. We can now state the central theorem of the representation theory in more detail:

GNS Construction

Let $\mathcal{A}$ be a C^* algebra, and let f be a state on $\mathcal{A}$. Then there exists a complex Hilbert space $\mathcal{H}_f$ with inner product $\langle .,. \rangle$, and a representation π_f of $\mathcal{A}$ in $\mathbf{B}(\mathcal{H}_f)$ with cyclic vector Φ such that $f(a)= \langle \Phi, \pi_f(a)\Phi \rangle$ for all $a \in \mathcal{A}$. Every representation of $\mathcal{A}$ can be obtained in this way. Further, if f is pure then π_f is irreducible.

What is involved in the GNS construction is that the Hilbert space $\mathcal{H}_f$ is constructed as the completion of a certain quotient space of $\mathcal{A}$ itself, by means of a state f on $\mathcal{A}$ (which state we say *generates* the representation). The quotient space is defined as the space of equivalence classes $a_f \subseteq \mathcal{A}$, where a,b are in a_f if $f(c^*(a-b))=0$ for all $c \in \mathcal{A}$ (that is, if a and b differ by an element in the left ideal of f , which is easily seen to be a linear subspace of $\mathcal{A}$).

The essential point is that the set of all these equivalence classes is itself a linear vector space with the non-degenerate inner product $\langle a_f, b_f \rangle = f(a_f^* b_f)$ (which is obviously sesquilinear since * is an anti-isomorphism on $\mathbb{C}$). This space, denote $\mathcal{H}_f$, is therefore pre-Hilbert. Completing in norm, we then define the action of $b \in \mathcal{A}$ as $ba_f = (ba)_f$

(this definition is independent of the particular a in a_f that we use). It is now obvious that the vector $\mathbb{1}_f$ in $\mathcal{H}_f$ is cyclic, because $a\mathbb{1}_f=(a\mathbb{1})_f=a_f$ so that $\mathcal{A}$ applied to the vector $\mathbb{1}_f$ yields every vector in the pre-Hilbert space; obviously completion of the set $\mathcal{A}\mathbb{1}_f$ is $\mathcal{H}$ so that this set is dense in $\mathcal{H}_f$.

2.3.9. The equivalence of representations.

From the GNS construction we obtain a large class of representations; too large, it might seem, since there is a representation π_f for each state f in $\mathcal{S}$. The first task is to establish the possible equivalences amongst these representations.

To begin with note that every vector ϕ in a representation π determines a state f_ϕ on $\mathcal{A}$ as the map $\mathcal{A} \longrightarrow \mathbb{C}$ given by $\langle\phi, \pi(a)\phi\rangle$. It is easy to show that this representation is then unitarily equivalent to the representation π_{f_ϕ} when ϕ is cyclic; that is there exists an isomorphism U: $\mathcal{H} \longrightarrow \mathcal{H}_{f_\phi}$, with $\pi_{f_\phi} = U\pi U^{-1}$. This means that given two representations π, π', if there is a cyclic vector in each which define the same state on $\mathcal{A}$ then these representations are unitarily equivalent. In an *irreducible* representation every vector is cyclic; in this case if *any* vector in $\mathcal{H}$ coincides with a vector in $\mathcal{H}'$ (in the sense that they define the same state on $\mathcal{A}$) then these representations are unitarily equivalent. Therefore the states associated with inequivalent irreducible representations belong to disjoint subsets of $\mathcal{S}$. (From now on we shall think of a vector state ϕ defined by a representation also as a state in $\mathcal{S}$; that is, we shall not distinguish f_ϕ from ϕ, and use the same symbol ϕ for both.)

We formalize these ideas as follows. We denote by $\mathfrak{h}_\pi$ the set of all vector states of a representation π (note that this includes states given as density matrices on $\mathcal{H}_\pi$; if we need the distinction between pure states and impure states we shall make it explicit). We shall also use the notation $\mathfrak{h}_f$ (meaning $\mathfrak{h}_{\pi_f}$); from the foregoing $\mathfrak{h}_\pi$ can be looked on as a

subset of $\mathcal{G}$. Given two cyclic representations π, π' and any two cyclic vectors ϕ, ϕ' in $\mathcal{H}, \mathcal{H}'$ respectively then the following three conditions are equivalent:

(i) $\phi \in \mathfrak{h}_{\pi'}$; $\phi' \in \mathfrak{h}_{\pi}$

(ii) π and π' are quasi-equivalent.

(iii) $\mathfrak{h}_{\pi} = \mathfrak{h}_{\pi'}$.

Despite this theorem there can still be a very large number of inequivalent irreducible representations. When $\mathcal{A}$ is given as the concrete C^*-algebra $\mathbf{B}(\mathcal{H})$ on some fixed Hilbert space $\mathcal{H}$, although every vector state is cyclic so that the representations generated by the vectors of $\mathcal{H}$ are all unitarily equivalent, there are still many states on $\mathbf{B}(\mathcal{H})$ which are not contained in any $\mathfrak{h}_{\phi}$, with $\phi \in \mathcal{H}$. There are actually 2^c of them, where c is the cardinality of the continuum, and those are just the irreducible ones.

None of them are faithful, however (the set of all *compact* operators in $\mathbf{B}(\mathcal{H})$ lies in the kernal of all of them (Kadison [1967 p.81]); equivalently any state on $\mathbf{B}(\mathcal{H})$ which is not a vector state assigns zero measure to every compact operator). A fundamental feature of the algebraic approach is that in more physical examples one actually deals with an algebra $\mathcal{A}$ which is *smaller* than $\mathbf{B}(\mathcal{H})$ (not every self-adjoint bounded operator is an observable). In that case there are inequivalent irreducible *faithful* representations.

One of the chief motivations for the application of these methods in quantum field theory was the hope that one might find some other characterisation of equivalence by which (at least in physically important cases) all these different representations can be subsumed. (As we have already remarked the existence of inequivalent representations automatically presents itself in QFT on independent grounds.)

This hope arose as a result of a theorem due to Fell [1960] who showed that for a C^*-algebra $\mathcal{A}$ which is a sub-algebra of $\mathbf{B}(\mathcal{H})$ one can nevertheless approximate any state on $\mathcal{A}$ by a

sequence (more properly a *net*[8]) of vector states in $\mathcal{H}$. Of course this approximation is not convergence in norm (for then the state approximated would lie in $\mathcal{H}$); it is *convergence in the weak * topology*. We earlier made passing reference to this topology; let us consider it in more detail.

Consider a topological linear space R and consider its dual, R^*, the set of all continuous linear maps to the reals. Then the elements of R are also linear maps on R^* (R^* has the obvious linear space structure $(f+\lambda g)(a)=f(a)+\lambda g(a)$). The weak * topology is the weakest topology with respect to which all these maps are continuous. And equally, R^* is closed with respect to this topology if it contains *all* continuous linear functionals on R (and it is the weakest topology with this property). When R is a C^*-algebra there is an obvious connection between the set of states $\mathcal{S}$ and R^* (namely that $\mathcal{S}$ is the set of all normalized positive elements in R^*); $\mathcal{S}$ is therefore closed in the weak * topology (this fact is used in the proof that the set of pure states is full on $\mathcal{U}$). The set of states $\mathcal{S}_\pi$ defined by a representation π is, on the other hand, closed in the *norm* topology (a stronger topology.) It is now obvious why $\mathcal{S}$ is larger than $\mathcal{S}_\pi$ in general; limit points in the weaker topology will not be limit points in the stronger topology.

The theorem of Fell above amounts to the fact that when $\mathcal{A}$ is a sub-algebra of $\mathbf{B}(\mathcal{H})$ the states given by $\mathcal{H}$ is large enough such that its weak * closure is $\mathcal{S}$. We would like to free this result from its dependence on $\mathcal{H}$; this can be done and one finds that if π is a *faithful* representation of a C^*-algebra then the weak * closure of $\mathcal{S}_\pi$ equals $\mathcal{S}$ (Emch [1972 Th 7 p.107]). If one bears in mind that there are physically significant algebras (for example the algebra of the canonical anticommutator relationships) which are simple (no two-sided closed proper ideals) and that the kernel of a

[8] The distinction is related to the question of whether or not the algebra is first countable. We shall not go into the details (see, e.g. Choquet [1969 Sec 4]).

representation is always such an ideal one sees that in these cases *every* state can be approximated by a sequence of states in *any* non-zero representation. More generally one might argue that non-faithful representations have no fundamental significance. In this way one obtains a kind of uniqueness theorem.

Representations in which the weak * closure of the vector states coincide are called **weakly equivalent.** Because of the operationalist interpretation of weak * convergence (see below) they are also called **physically equivalent.** It was soon to become clear that "physically equivalent" but unitarily inequivalent representations *do* have physical applications, so this terminology is not very fortunate. The initial hope that it is sufficient, in physics, to deal with weakly equivalent representations has therefore been frustrated.

Nevertheless, it is possible to give an operational interpretation of weak * convergence: this follows from the observation that the weak * topology has a basis of neighbourhoods obtained by considering *all* sets of the form $N(f,\mathcal{F},\varepsilon) = \{g\in\mathcal{G};\ f(a)-g(a) < \varepsilon \text{ for all } a\in\mathcal{F}\}$ where $\mathcal{F}$ is a *finite* subset of $\mathcal{A}$ (or of $\mathcal{U}$). This means that weak * convergence of a sequence $\{f_i\}$ of states on a state f takes the form: given any *finite set of observables* $\mathcal{F}$, and any $\varepsilon>0$, there exists an integer $n(\mathcal{F},\varepsilon)$ such that $f_i(a)-f(a) \leq \varepsilon$ for every $i\geq n(\mathcal{F},\varepsilon)$ and for every $a\in\mathcal{F}$. In operational terms one can motivate the use of this approximation in terms of the *limitations of laboratory experiments*; since one can only measure a finite number of observables it is always sufficient, from a phenomenological point of view, to limit oneself to a finite set of observables in the use of approximations.

We shall return to these ideas in Section 3.5; they play an important rôle in the measurement theory that is there developed. For the moment we anticipate that material and state that the key issue is that of perspicuity or convenience. Whilst it may be true that in any (faithful)

representation there will be a vector state which, from the point of view of any conceivable experiment or set of experiments conducted in finite times, will accurately deescribe Ω to any required accuracy, it does not at all follow that such a state will present itself in a straightforward way or that we can determine the measures or possessed values of the observables that we are interested in. Depending on the phenomenology that we wish to describe it may be that a different representation will be better fitted to this end.

2.3.10. Von Neumann algebras.

The foregoing has an intimate relationship to the theory of von Neumann algebras; in the weak * topology we have something very like the weak operator topology that von Neumann introduced in 1936 and in his subsequent collaboration with Murray.

Let π be a representation of a C^*-algebra; we denote by $\pi(\mathcal{A})'$ the **commutant** of $\pi(\mathcal{A})$, that is the subset of $B(\mathcal{H})$ which commutes with all elements in $\pi(\mathcal{A})$ (so that elements in the commutant need not necessarily be in $\pi(\mathcal{A})$). The **bicommutant** $\pi(\mathcal{A})''$ is the commutant of $\pi(\mathcal{A})'$ (so that the same caution applies). A **von Neumann algebra** $\mathcal{A}$ is a sub-algebra of $B(\mathcal{H})$ such that $\mathcal{A}=\mathcal{A}''$. As one might expect, there is a way of associating every C^*-algebra with a von Neumann algebra (thus freeing it from any particular choice of Hilbert space). We shall come on to this in a moment. Note that if a representation π is irreducible then $\pi(\mathcal{A})'$ is trivial ($\mathbb{0}$ and $\mathbb{I}$ alone) so its bicommutant is always $B(\mathcal{H}_\pi)$.

One of the most important properties of a von Neumann algebra is that it is generated by its projections, in the sense that $\mathcal{A}=\mathcal{L}''$. For this reason the classification of von Neumann algebras proceeds from their associated projection lattices. This is related to the topology of a von Neumann algebra; whilst it is indeed closed in the norm topology (so in particular it is a C^*-algebra) it is also closed in a number of weaker topologies (the von Neumann density

theorem). To see what is involved, we need to define two of them. The **ultraweak topology** is the weakest topology for which the maps $B \longrightarrow \Sigma_i \langle \phi_i, B\psi_j \rangle$ are continuous (with $\Sigma_i |\phi_i|^2$ finite.) The **ultrastrong topology** is the weakest topology for which the maps $B \longrightarrow \Sigma_i B\psi_i \in \mathcal{H}$ are continuous (relative to the norm topology on $\mathcal{H}$). Both are weaker than the norm topology, and a von Neumann algebra is closed with respect to both of these topologies. Closure with respect to the ultraweak topology is equivalent to the "sequential weak closure" used by Plymen and Davies in their construction of Σ^*-algebras; the following includes a Hilbert space version of the basic fact used by Plymen that we referred to in (2.3.6).

Let $\mathcal{A}$ be a von Neumann algebra which is a sub-algebra of $\mathbf{B}(\mathcal{H})$; for any state f in $\mathcal{G}$ the following conditions are equivalent:

(i)f is continuous on $\mathcal{A}$, the latter being equipped with the ultraweak topology.

(ii)f is continuous on $\mathcal{A}$, the latter being equipped with the ultrastrong topology.

(iii)There exist $\{\phi_i\}, \{\psi_j\}$ with $\Sigma_i \|\phi_i\|^2$, $\Sigma_j \|\psi_i\|^2$, finite, $\phi_i, \psi_j \in \mathcal{H}$, such that $f(a) = \Sigma_i \langle \phi_i, a\psi_i \rangle$ for every $a \in \mathcal{A}$.

(iv) $f(\Sigma_i p_i) = \Sigma_i f(p_i)$ for every family $\{p_i\}$ of mutually disjoint projections in $\mathcal{A}$.

(For proof see Emch [1972 p.118]; a state on a von Neumann algebra which satisfies any one of these continuity conditions is called **normal**).

Condition (iv) is a mild strengthening of the familiar requirement that the states be *countably additive* on the projection lattice (when $\mathcal{H}$ is separable every family of mutually orthogonal projections is countable). In the separable case, when $\mathcal{A} = \mathbf{B}(\mathcal{H})$, the relationship (iii) $\Leftrightarrow$ (iv) is Gleason's theorem. It is now obvious why Gleason's theorem does not hold for a general C^*-algebra; the stronger the topology on the domain of the states, the less constrained the maps become through the requirement of continuity, because there are fewer sequences of observables on which they have to converge. The norm topology on a

C^*-algebra is so strong that the states are relatively wild.

The **universal enveloping von Neumann algebra** $\mathcal{A}''$ of a general C^*-algebra $\mathcal{A}$ is defined in a completely intrinsic way, in contrast to the foregoing. Essentially we take the GNS representation of every state and paste them all together; that is we form $\mathcal{H}=\oplus_{f\in\mathfrak{S}}\mathcal{H}_f$ and define $\pi:\mathcal{A}\longrightarrow\mathbf{B}(\mathcal{H})$ by the requirement that for each $a\in\mathcal{A}$ $\pi(a)$ is that element of $\mathbf{B}(\mathcal{H})$ which induces $\pi_f(a)$ on $\mathcal{H}_f$. If we now form the algebra $\pi(\mathcal{A})''$ we know that it is von Neumann; by construction it is faithful. It is also clear by construction that to every state $f\in\mathfrak{S}$ there is a vector state in $\mathcal{H}$; conversely every vector state in $\mathcal{H}$ is a state in $\mathfrak{S}$ (since π is faithful). We conclude that $\mathfrak{S}=\mathfrak{h}_\pi$; in other words, the normal states of the universal enveloping von Neumann algebra of a C^*-algebra exhaust all the states in $\mathfrak{S}$.

This result is important. It tells us that there always exists a representation of a C^*-algebra in which every state is given as a vector state. In this way the standard assumptions of quantum theory are secure, *viz*, that for every physical system Ω there is a set of self-adjoint operators acting on a Hilbert space, and every state is normal. The trouble is, the space is so very large, and so highly reducible. There is no question of making this representation concrete.

We shall describe the classification theory of von Neumann algebras in (2.4.8); as we shall see, it plays an essential rôle in the Wigner classification of representations of the inhomogeneous Lorentz group (Section 3.1) and in the general representation theory of quantum fields (*cf.* (3.4.9) and Section 3.5).

2.3.11. Koopman Systems.

We conclude this section with one more look at the classical case, that is, the representation theory for associative $\mathcal{U}$ (equivalently commutative $\mathcal{A}$). The representation that we wrote down earlier is a faithful one, but several typical features of the GNS construction are not apparent. It is helpful to examine the so-called **Koopman systems**, named after their discoverer (Koopman [1931]). We recall from the previous theorem that to every state $f\in\mathcal{S}$ there exists a regular Borel measure μ on a space X. Now let $L^2(X,\mu)$ be the complex Hilbert space of square integrable functions $X \longrightarrow \mathbb{C}$ (with respect to the measure μ). For every $a\in\mathcal{U}$ we define the bounded self-adjoint operator $\pi_f(a)$ on this L^2 space by $(\pi_f(a)\psi)(x) = A(x)\psi(x)$, where $A(x) = (\pi_f(a))(x)$ is the function on X determined from Theorem 2.3.1. It is easy to see that this is a representation of $\mathcal{U}$ as a (commutative) sub-algebra of $B(L^2(X,\mu))$. We also note that the vector Φ defined by $\Phi(x)=1$ for all $x\in X$ satisfies $\langle\Phi,\pi_f(a)\Phi\rangle = \int A(x)d\mu = f(a)$; it is also *cyclic*. The Koopman system is the canonical classical parallel of the GNS construction.

We consider once more the question of unitary equivalence of representations. We recall the result stated above: two representations π,π' are unitarily equivalent if and only if $\mathfrak{h}_\pi, \mathfrak{h}_{\pi'}$ have some *cyclic* state in common. A sufficient condition for this to be so is that there exists a *unitary* element U in $\mathcal{A}$ (so that $UU^*=U^*U=\mathbb{I}$) such that for some pair of cyclic vectors f,f' in $\mathfrak{h}_\pi, \mathfrak{h}_{\pi'}$ the condition $f(a) = f'(U^*aU)$ holds for every $a\in\mathcal{A}$. Such a $U\in\mathcal{A}$ defines an (inner) automorphism of the algebra; recall from the GNS construction that if $\xi(a)$ is the equivalence class of a corresponding to a vector in $\mathcal{H}_f$ then $b\xi(a)=\xi(ba)$, $\xi(\mathbb{I})=\phi$ is the generating vector of the representation, and $\langle\phi,\pi(a)\phi\rangle = f(\mathbb{I}^*a\mathbb{I})=f(a)$. Any other vector in $\mathcal{H}_f$ can be approximated (in norm) as closely as we like; that is, for any $\psi\in\mathcal{H}_f$ and $\varepsilon>0$ there is some unitary $U\in\mathcal{A}$ such that $\langle\psi,\pi(a)\psi\rangle - \langle\phi,\pi(U^*aU)\phi\rangle \leq \varepsilon$ for all $a\in\mathcal{A}$. Clearly if ε can be chosen as zero we have the condition above, so that this condition is sufficient but not necessary for the unitary equivalence of

two representations. In fact necessity follows when the cyclic vectors are *pure*. That is, if two representations $\pi_f, \pi_{f'}$ are unitarily equivalent and f,f′ are pure then there exists unitary $U \in \mathcal{A}$ such that $f(a)=f'(U^*aU)$ for all $a \in \mathcal{A}$ (Kadison [1967 p.75]).

With this in mind we see that when $\mathcal{A}$ is commutative then for any unitary $U \in \mathcal{A}$ and for any $a \in \mathcal{A}$, $U^*aU = a$; that is there cannot exist any f,f′ (pure or otherwise) such that $f(a)=f'(U^*aU)$ unless f and f′ are identical. If now f,f′ *are* pure (and distinct) their associated representations are never unitarily equivalent. In particular all *irreducible* representations of a classical system are unitarily inequivalent. Consider again the Koopman system. A pure state has measure concentrated on a single point of X; it assigns possessed values to *every* observable in $\mathcal{U}$. Therefore two classical systems for which a single observable has distinct possessed values (and for which every observable *has* a possessed value) are unitarily inequivalent.

This means that one cannot unitarily implement the dynamics of a classical system (with unitary operators *in the algebra* $\mathcal{A}$); but there may be unitary operators in $\mathbf{B}(\mathcal{H})$ which implement the dynamics, and there may be automorphisms of the algebra which are not unitarily implemented. For the Koopman system we have the former alternative. Later we shall see that in the really crucial case when Ω has both classical and quantum observables the exact classical observables cannot change under *any* automorphic evolution (*cf.* the concluding remarks of the previous section).

Before we conclude let us check that we have no inconsistency between these conclusions and the fact, for example, that each cyclic vector in a cyclic representation gives rise to a state in $\mathcal{S}$ whose corresponding representation is unitarily equivalent to the given representation. For a Koopman system generated by a pure state, we have an irreducible representation, so that every state given by this representation is cyclic, so we should

have unitary equivalence with every such state, and yet as we have seen we do not have any equivalence with another irreducible representation.

The solution to this conundrum is that when $f \in \mathcal{G}$ is pure the Koopman system is rather pathological; all the vector states in $\mathfrak{h}_f$ give rise to the *same* state in $\mathcal{G}$, namely f, because every L^2 function on X with respect to a measure concentrated on a single point p (recall that for f pure $d\mu(x)=\delta(x-p)dx$ for some $p \in X$) and normalized to unity with respect to this measure must statisfy $\phi(p)=1$; consequently $\langle \phi, \pi(a)\phi \rangle = A(p)$ for every $\phi \in \mathfrak{h}_f$ and the state thus determined on $\mathcal{U}$ is f. The Koopman systems are only useful in statistical mechanics, as representations given by impure f; they are then determined (up to unitary equivalence) by the measure class of the associated Borel measure.

These ideas will be taken up in our discussion of measurement theory. The first axiomatization of quantum field theory by algebraic methods made use of the properties of von Neumann algebras. The so-called W^*-algebras have also been useful in making rigorous the comments above on the relationship between the Segal and Mackey theories (Plymen [1968a,b]) (a W^*-algebra is a C^*-algebra which is the dual of a Banach space), and there are other variants besides the Σ^* and W^*-algebras (Deliyannis [1975]); W^*-algebras have their champions, from a philosophical basis also (Primas [1980]). All of these algebras are, however, C^*-algebras. We shall review the Haag-Kastler C^*-algebra postulates for a general QFT in Section 2.5.

In subsequent sections we shall denote by $\langle ; \rangle$ the canonical duality between $\mathcal{G}$ and $\mathcal{A}$, i.e. $f(a) = \langle f; a \rangle$ for all $f \in \mathcal{G}, a \in \mathcal{A}$.

2.4 The Representation Theory of Groups

Imprime, *v.* 1575: to begin
The Shorter Oxford English Dictionary

2.4.1 Introduction.

The suggestion that quantum mechanics is very largely about the representation of spacetime groups and subgroups was first made by Hermann Weyl in 1927; some twenty years later, and following the intensive development of Hilbert space representation theory at the hands of von Neumann, Wigner, Stone, Gel'fand and Mackey (to name the principal architects of the theory), this suggestion had been fully vindicated. At the same time, this theory made significant inroads in a fundamental problem of pure mathematics: to find and classify the representations of non-compact, non-abelian groups. In this story we see the most profound and beautiful interplay between physics and mathematics.

I cannot do justice to the full ramifications of this story, but I shall try and summarize those parts which bear directly on quantum theory culminating in the imprimitivity theory of Mackey[1]. Although the Wigner classification of representations of the Lorentz group is a step in the development of this theory, I shall not discuss this

[1]The literature on the history of group theory is enormous; our interest in this section is the exegesis of the fundamental ideas of the imprimitivity theory, and to indicate the natural role of Hilbert space theory in the representation theory of groups; if familiarity with this material may be assumed, this section may be omitted altogether, or rather replaced by a single reference: Mackey's unmatched historical survey of the subject, to be found in Mackey [1980].

classification at this stage but in the following chapter. Neither shall I at this stage discuss the connections of the representation theory with the fundamental model; this will be done in the following section. For the moment it will suffice to know that if the action of a continuous symmetry group on $\mathcal{A}$ is automorphic in a representation which admits a (cyclic) vacuum, it follows that the symmetry group must be implemented by unitary operators; in particular the relevant representation is a strongly continuous unitary representation on a complex Hilbert space.

2.4.2. Historical background.

Group theory, as a fully accessible branch of mathematics, is just over a hundred years old. Its beginnings lie not so much in the idea of spacetime transformations, however natural and intuitive this idea now appears, but in the groups of finite permutations. The reasons are pragmatic: group theory developed not because of its *a priori* elegance or mathematical beauty, but because of its importance in the theory of real polynomials: first at the hands of Lagrange (polynomials of second, third and forth order), and later Abel and Ruffini (who most particularly sought to understand the obstacles to extending Lagrange's methods to fifth and higher order equations). Cauchy was the first to make the rôle of permutation groups explicit in this connection and to formulate some basic definitions (of subgroup, transitive group, and conjugate elements); he also first attempted to classify these groups.

The significance of the Gal-ois theory to the emergence of the group concept is well known (the term group is due to him, as also the notions of a normal subgroup and quotient group). Following the much-delayed publication of this theory in 1847 first Kronecker, and then Cayley, made important contributions to the theory of elliptic functions and modular forms. But it was not until Sophus Lie initiated the study of continuous groups that the theory became more than the preserve of a few specialists. From the 1870's onwards, most especially with Felix Klein's popularization of these ideas, the group concept was recognized as pivotal

throughout mathematics, with applications in geometry, real and complex analysis, and number theory, as well as the more specialized topics of ellipitic functions and algebraic equations. At this stage the understanding of its relevance to physics was still in an embryonic stage.

At this time - that is in the early 1880's - the notion of a group *representation* was still obscure; rather a different tool in the analysis of groups was developed, namely the idea of a group *character.* Weber was the first to define this idea for abstract finite commuative groups, although it played an implicit rôle in Gauss's work on the theory of binary quadratic forms at the turn of the century, as also in Dedekind's ideal theory for algebraic number fields. Of course, from a more modern viewpoint, the notion of group characters also underlies Fourier analysis, as we shall see in a moment. Weber defined a character as a complex valued function χ on a (finite commutative) group G such that $\chi(xy) = \chi(x)\chi(y)$ for all $x,y \in G$; this idea was shortly afterwards used by Dedekind in the following connection: let G be a finite group of order h and let $g_1 \ldots g_h$ be its elements. If $x_{g_1} \ldots x_{g_h}$ are h independent variables parameterized by the elements of G then $\chi(x_1 \ldots x_h) = \det x_{g_i g_j^{-1}}$ is a polynomial in h variables (the "group determinant"). Dedekind discovered that χ factorizes into linear factors parameterized by the characters of G whenever G is commutative, and for certain non-commutative groups, but he could not obtain a general result along these lines in the non-commutative case. It was at his urging that Frobenius became interested in the problem. Whilst his first (successful) analysis used a rather complicated definition of character, he shortly thereafter reworked the theory using the idea of a group *representation*, and achieved a much simpler formulation. An n-dimensional representation R of a finite group G is a homomorphism of G into the group of all $n \times n$ invertible matrices with complex entries; it is *reducible* if there exists a basis such that for each $x \in G$, R_x has the form $\begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$. A character χ^R on G can then be defined as $\chi^R(x) = \mathrm{Tr}(R_x)$. It is then clear that the characters of

Dedekind and Weber are the characters of the *one-dimensional* representations of G; further A_x and C_x (as matrix homomorphisms of G) are also representations, and Frobenius showed that their characters are simply related to the character of R as: $\chi^R(x) = \chi^A(x)+\chi^C(x)$. It also follows that the definition of character is basis independent (through the independence of the trace) and that a character is constant on the conjugacy classes of the group; further two distinct irreducible characters (*i.e.* characters of irreducible representations) are *orthogonal* in the sense that $\sum_{x\in G} \chi^1(x)\overline{\chi^2(x)} = 0$. Therefore irreducible characters are linearly independent so they are finite in number (less than h); he showed there must be exactly h of them, and actually determined them all for a number of non-commutative groups.

In a series of papers at the close of the century Frobenius gave the essentially complete structure theory of finite dimensional groups; from our point of view of particular importance was his introduction, in 1898, of the idea of an *induced character*. Given a normal subgroup H of a group G and a character χ, define χ_H as the restriction of χ to H and define χ' by $\chi'(x) = h^{-1}\sum_{y\in G} \chi_H(yxy^{-1})$ Then χ' is also a character (*induced* by χ). The power of this idea is shown, for example, by the fact that for nilpotent groups *every* irreducible character which is not one-dimensional is induced by a one-dimensional character (for some subgroup H). This concept was generalized by Mackey to yield the inducing construction, which will play a major rôle in the representation theory of the spacetime groups.

The importance of the idea of group representations was further emphasized by Burnside, who showed that reducible representations (in the sense defined above) can actually be put into diagonal form, and cast the reduction theory into the familiar form in which any representation can be written as the direct sum of irreducible representations.

A representation of fundamental importance that also emerged

at this time is the *regular representation*. Consider the space V of all complex valued functions of a group G and the representation: $R_x: V \longrightarrow V$ given by $R_x(f)(y) = f(yx)$ (that is, the representation is given by translation). The importance of this representation is clear from the theorem that the regular representation can be written as a direct sum in which each irreducible representation occurs with a multiplicity equal to the dimension of the space V. Consider now the infinite dimensional case; it is clear that a simple extrapolation of the Frobenius theory would lead to an infinite number of irreducible characters and that the analogue of the regular representation would have to use a generalization of the direct sum which extends to the infinite case. It is also clear that one will have to use some sort of measure over the group to replace the factor h^{-1}, which indicates that *compact* groups may offer the simplest infinite dimensional extension of the finite theory. The essential step was taken by Schur in 1924; adapting a group integral constructed by Hurwitz for the rotation group he was able to take over the principal features of the Frobenius theory to this type of (infinite, compact) group. Shortly after Haar showed that any locally compact group must admit such a measure (which is defined on all Borel sets and invariant under right translation). Von Neumann then proved uniqueness. The generalization of Schur's results to arbitrary semi-simple Lie groups was made by Weyl soon after. By 1927 Weyl was in possession of the central connection with Fourier analysis and classical analysis; he also saw that these ideas underlay significant parts of the nascent quantum theory.

2.4.3 The connection with Fourier analysis.

Although it was only with the Peter-Weyl theorem that the group theoretic foundation of Fourier analysis became obvious, one can already see striking similarities between certain formulas that occur in the finite theory, and the Fourier transforms

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx} \quad , \quad a_n = (2\pi)^{-1}\int_0^{2\pi} f(x)\, e^{-inx}dx.$$

Consider now the regular representation of a finite *commutative* group; since this representation is the direct sum of the irreducible representations associated with each character, and in the commutative case the irreducible characters are all one-dimensional, it follows that every complex valued function f on G is uniquely given by $\Sigma a_\chi \chi(x)$ (the sum is over all the characters of G, denote $\hat{G}$). From the orthogonality relationships between the characters we also have that:

$$a_\chi = (h)^{-1}\sum f(x)\ \overline{\chi(x)}dx.$$

In terms of the discrete measure μ this may be written

$$a_\chi = (\mu(G))^{-1}\int_G f(x)\overline{\chi(x)}d\mu,$$

and noticing that the characters of the additive group on the reals modulo 2π are precisely the functions e^{inx} the correspondence between the two sets of formulae becomes clear. In particular one sees that one has a correspondence between the Fourier transform space and the space of characters of G, that is the dual space $\hat{G}$, and that translation of functions on G goes over to multiplication of functions on $\hat{G}$.

The Peter-Weyl theorem makes this correspondence precise. It tells us that given a set of matrices L which is an irreducible representation of a group G then the matrix elements span a vector space V, independent of the basis chosen, and invariant under right and left translations. For each inequivalent representation one obtains a vector space and when the group is a compact Lie group, these spaces are all orthogonal with respect to Haar measure. Taken together, their linear span is dense in the space of continuous

functions on the group; therefore the set of basis vectors which is orthogonal restricted to each of the vector spaces V defines a complete system of functions on the group. This is just the situation above; it is the statement that functions of period 2π (which are therefore defined as functions on the one-dimensional torus group) may be expanded in a Fourier series in the functions e^{inx}, which are the continuous characters on the group.

It is obvious that the space of characters $\hat{G}$ contains much of the structure of G; just how much was made clear by Pontrjagin in 1934 and van Kempen in 1935, in the theory of *group duality*. Obviously for any G the set $\hat{G}$ is a group under multiplication of characters ($\chi_1(x)\chi_2(x)$ is a character, because $\chi_1(x)\chi_2(x)\chi_1(y)\chi_2(y) = \chi_1(x)\chi_1(y)\chi_2(x)\chi_2(y) = \chi_1(xy)\chi_2(xy)$); further for each χ in $\hat{G}$ the map $x: \chi \longrightarrow \chi(x)$ is a complex-valued function on $\hat{G}$ and is hence a character on $\hat{G}$, *i.e.* a member of $\hat{\hat{G}}$. When G is a compact commutative *countable* group Pontrjagin showed that in the topology on $\hat{G}$ such that all these characters on $\hat{G}$ are continuous, $\hat{G}$ is compact and separable. It is of course also a commutative group. He also showed the converse, that for any compact separable commutative topological group A, its continuous characters form a countable (commutative compact) group $\hat{A}$; Pontrjagin duality now asserts that the groups $\hat{\hat{A}}$ and A , and likewise $\hat{\hat{G}}$ and G, are isomorphic; in the former case they are also homeomorophic. van Kampen generalized this statement to arbitrary locally compact commutative groups, and showed that the double dual of any such group G is actually homeomorphic to G. The dual $\hat{G}$ is always a locally compact commutative group.

It is at this stage that Hilbert space theory enters the picture, in particular following the discovery of Haar measure in 1933 and the realization that the fundamental mathematical statements of quantum mechanics have a very simple meaning concerning *unitary representations* of the additive group. Hilbert space theory, group duality, and the Peter-Weyl theorem combined to give a remarkable synthesis

which yields analogues of the spectral theorem, the Riesz-Fischer theorem, the Plancherel theorem, and the Bochner-Herglotz theorem in group theoretic terms, as well as the classical harmonic analysis already contained in the Peter-Weyl theorem. With these tools it is a simple matter to prove, for example, the von Neumann uniqueness theorem on the essential uniqueness of the CCR's, as a consequence of the essential uniqueness of the direct sum decomposition into irreducibles of the regular representation of the additive group of the reals.

2.4.4. The introduction of Hilbert space theory.

There is a fundamental relationship between the spectral theorem of von Neumann and Hilbert and the theory of *unitary* representations of the reals. This insight is due once more to Weyl, although it was Stone who first gave the details of the equivalence. Recall that the spectral theorem of Hilbert states that for every Hilbert-Schmidt operator A on a separable Hilbert space $\mathcal{H}$ there is a unique projection-valued measure P on the reals with bounded support such that $(f,Af) = \int x\, d(f,P_x f)$, and conversely. Von Neumann extended this theorem to unbounded operators through his theory of essential self-adjointness.

Weyl noticed that for any bounded self-adjoint operator A the operator $U_t = e^{iAt}$ defined by its power series expansion is unitary, and a representation of the additive group of the reals. Moreover he saw that this representation is *continuous* in the sense that $t \longrightarrow U_t f$ is a continuous $\mathcal{H}$-valued map (with respect to the norm topology on $\mathcal{H}$). Therefore every projection-valued measure on the reals with bounded support defines a continuous unitary representation of the additive group of the reals, with a bounded self-adjoint operator as generator. Weyl now suggested that the von Neumann extension of the spectral theorem should make it possible to associate continuous unitary representations of the reals with arbitrary projection-valued measures. This was proved by Stone in 1930: he showed that they stand in one-one correspondence. As a result there is also a one-one correspondence between

self-adjoint operators and continuous unitary representations of the additive group of the reals, and "diagonalizing" the operator A is equivalent to decomposing the group representation as a "direct sum"; the quotation marks may be removed when the group is compact, otherwise one needs the spectral theorem on the one hand, and a generalization of direct sum to some sort of integral on the other.

Let us now see how group duality and the Fourier transform are related to Hilbert space theory. These relationships were first made explicit by André Weil in his book "L'integration dans les groupes topologiques et ses applications a l'analyse" in 1938, but they were fairly straightforward consequences of the ideas discussed above.

With μ as Haar measure on a locally compact commutative group G define $L^2(G,\mu)$ as the space of square integrable functions on G (with respect to the measure μ) complete in the L^2 norm. The Fourier transform of $f\in L^2(G,\mu)$ is defined as $\hat{f}:\hat{G} \longrightarrow \mathbb{C}$ given by $\hat{f}(\chi) = \int \chi(x)f(x)d\mu(x)$ for all $\chi \in \hat{G}$. When f is also integrable $\hat{f}$ is continuous and square integrable and completing the space $L^2(\hat{G},\hat{\mu})$ gives a Hilbert space. $\hat{}$ is norm-preserving on the (dense) domain of f in $L^2 \cap L^1$ onto the (dense) range of f in $L^2 \cap \mathcal{C}(\hat{G})$ and hence extends to an isomorphism of $L^2(G,\mu)$ with $L^2(\hat{G},\hat{\mu})$ (the generalized Plancherel theorem).

Now let us make the connection with the spectral theorem. We do so by way of the Bochner-Herglotz-Weil theorem: define for any finite measure μ on $\hat{G}$ the transform $\hat{\mu}$ as $\hat{\mu}(x) = \int\chi(x)d\mu(\chi)$, and notice that $\hat{\mu}$ is positive definite: that is, $\sum c_i \bar{c}_j \hat{\mu}(x_i x_j^{-1}) \geq 0$ for all pairs of n-tuples $(c_1,\ldots,c_n),(x_1,\ldots,x_n)$, $x_i\in G$, $c_i\in \mathbb{C}$. From the foregoing, $\hat{\mu}$ must be continuous; therefore the Fourier transform of any finite measure on $\hat{G}$ is a continuous positive definite function on G. Weil proved that *every* continuous positive definite function f on G must be given by such a measure μ on $\hat{G}$, such that $f = \hat{\mu}$. The connection with the spectral theorem becomes clear when one notices that if U_x is a

continuous unitary representation of any topological group in a Hilbert space $\mathcal{H}$ then $(f,U_x f)$ is a continuous positive-definite function on the group (because $\sum c_i \bar{c}_j (f, U_{x_i x_j^{-1}} f) = (\Sigma_i c_i U_{x_i} f . \Sigma_j c_j U_{x_j} f) \geq 0$). In the case of the additive group of the reals the spectral theorem says that there is a one-one correspondence between continuous unitary representations U of G and projection-valued measures P on $\hat{G}$ such that $(f,U_x f) = \int \chi(x)\, d(f,P_\chi f)$; it therfore leads directly to the Bochner-Herglotz-Weyl theorem, for those positive continuous functions on G which can be written in the form $(f,U_x f)$, with $\mu(E) = (f,P_E f)$ for E a Borel set in $\hat{G}$. As Gel'fand and Raikov later pointed out, every positive continuous function on G can be written in this form, so the two theorems are saying exactly the same thing. Incidentally, this is true also for any commutative locally compact group.

The results of these investigations had, by the end of the war, led to the fundamental structure theorem that the problem of obtaining all the continuous unitary representations of a locally compact commutative group is the same as the problem of obtaining all the projection valued-measures on its dual. It turns out that the latter problem does not depend on the group theoretic properties of $\hat{G}$ but only on its measure theoretic properties, that is, the Borel algebra of the set $\hat{G}$ that is used to define the properties of a measure. The next step is to establish the equivalence classes of such projection-valued measures and then to try and generalize as much as we can of the foregoing to the non-commutative case.

First, however, let us see how we may understand the CCR's in terms of these ideas. Like von Neumann's, Weyl's mathematical insights were closely connected with his study of quantum theory. Of course in quantum theory one had already a very simple and direct stimulus to study the theory of unitary representations of groups on Hilbert space, most particularly following Wigner's [1931] result on the (anti)unitarity of transformations which preserve the projective geometry of Hilbert space. Weyl essentially

extended the group theoretical insight to quantum theory, to include the "strange relation[2]" at the heart of the matrix mechanics - the CCR between position and momentum.

2.4.5. The canonical commutation relationships.

In 1927 Weyl noticed that the CCR's in the Schrödinger representation

$Q_i f(q_1,..q_n) = q_i f(q_1,..q_n)$, $P_i f(q_1,..q_n) = -i\hbar\frac{\partial}{\partial q_i} f(q_1,..q_n)$

with the Q's and P's the position and momentum operators on $L^2(\mathbb{R}^n, d^n x)$, lead to the following commutation relationships for the unitary operators $U^j(s)$, $V^i(t)$, defined by

$$V^j(t)f(q_1,..q_n) = e^{iQ_j t/\hbar} f(q_1,..q_n) = e^{iq_j s/\hbar} f(q_1,..q_n) \qquad (1)$$
$$U^i(s)f(q_1,..q_n) = e^{iP_i s/\hbar} f(q_1,..q_n) = f(q_1,..,q_i+s,....q_n)$$

for s,t in $\mathbb{R}$ (more suggestively, in the additive group of $\mathbb{R}$):

$[U^i(s_1),U^j(s_2)] = [V^i(t_1),V^j(t_2)] = 0;$

$U^i(s)V^j(t) = e^{ist\delta_{ij}/\hbar} V^j(t)U^i(s).$

If one now defines the unitary operators $U(\mathbf{s}),V(\mathbf{t})$ with $\mathbf{s},\mathbf{t} \in \mathbb{R}^n$ (or the additive group of $\mathbb{R}^n$) $U(\mathbf{s}) = U^1(s_1)...U^n(s_n)$, $V(\mathbf{t}) = V^1(t_1)....V^n(t_n)$, one obtains the commutation relationships:

$[U(\mathbf{s}_1),U(\mathbf{s}_2)] = [V(\mathbf{t}_1),V(\mathbf{t}_2)] = 0$, $U(\mathbf{s})V(\mathbf{t}) = e^{i(\mathbf{s}.\mathbf{t})/\hbar} V(\mathbf{t})U(\mathbf{s})$.

Clearly the U's and V's alone are unitary representations of the additive group in $\mathbb{R}^n$: $U(\mathbf{s}_1)U(\mathbf{s}_2) = U(\mathbf{s}_1+\mathbf{s}_2)$ (and likewise for V). On the other hand recognizing that $e^{i(\mathbf{s}.\mathbf{t})/\hbar}$ is the most general continuous character on the additive group G on $\mathbb{R}^n$, where a character χ_t is fixed by the choice of a vector $\mathbf{t}$ in $\hat{G}$ (recall that for finite dimensional vector spaces G and $\hat{G}$ are isomorphic), one can write this last relationship as:

$$U(\mathbf{s})V(\chi_t) = \chi_t(\mathbf{s})V(\chi)U(\mathbf{s}). \qquad (2)$$

[2] Born [1978 p.218]. Incidentally, Pauli's response to Born's attempt to distill the CCR's from Heisenberg's paper of [1925] (which he had correctly guessed) met with the following response from Pauli: "Yes, I know you are fond of tedious and complicated formalisms. You are only going to spoil Heisenberg's physical ideas by your futile mathematics" (Born [1976 p.218]). I dare not think of Pauli's response were he to read the present thesis.

If we think of t as an element of $\hat{G}$ and use the symbol χ accordingly, the group theoretic meaning of this relationship becomes clear. Define the unitary operator $W(s,\chi) = U(s)V(\chi)$ and evaluate the quantitiy $W(s_1+s_2,\ \chi_1\chi_2)$ to find

$$W(s_1+s_2,\ \chi_1\chi_2) = \chi_1(s_2)W(s_1,\chi_1)W(s_2,\chi_2);$$

this is a "representation up to phase" of the product group $G \times \hat{G}$, with the natural composition law $(s_1,\chi_1).(s_2,\chi_2) = (s_1+s_2,\chi_1\chi_2)$. Such a representation is called a *projective representation* and the phase factor which occurs in the composition law in the representation the *multiplier* of the representation.

As we have already mentioned ((1.4.6)), a proof was soon forthcoming (von Neumann [1931]) that Eq.(1) is the *unique* (up to unitary equivalence) representation of Eq.(2) . This uniqueness was not known to Weyl in 1927, but he seems to anticipate the conclusion which is then forced upon us: the CCR's of quantum mechanics are the infinitesimal form (the Lie algebra) of the projective representation of the group $G\times\hat{G}$. In some sense, Weyl suggested, the CCR's (and likewise the Schrödinger equation) can be *derived* from this elementary representation of symmetry.

The group $G\times\hat{G}$ can be identified with the *classical phase space* of an n-particle system; the commutation relationship for the W's multiplied by the factor $\chi(s)^{-1/2}$ then takes the form

$$[W(s_1,\chi_1),W(s_2,\chi_2)] = (\chi_2(s_1)/\chi_1(s_2))^{1/2}W(s_1+s_2,\chi_1\chi_2).$$

The phase factor can be associated[3] with the *symplectic form*

[3] For any locally compact topological group there is an equivalence relationship on the multipliers of a group and the quotient set that it generates is directly related to the homology of the group. In particular because the vector group on $\mathbb{R}^n$ is abelian and connected the multipliers are in correspondence with the skew-symmetric bilinear forms on $\mathbb{R}^n \times \mathbb{R}^n$. (Varadarajan [1968 Th. 10.38]).

on phase space. This connection, between the CCR's as an expression of symmetry, and as an expression of a quantization condition, will be taken up in (3.2.2) (see also (2.5.7)). The CCR's written in (any one of these) global forms are called the **Weyl relationships**.

2.4.6. Systems of Imprimitivity; heuristics

We now take up the programme of classifying the representations of a locally compact group, which as we have indicated is closely related to the classification of projection-valued measures on a suitable space. In (2.4.8) we review the inducing construction, in the special case of transitive systems, which was the essential tool of Wigner in his paper of 1939. The *general* theory is due to Mackey, in a series of papers from 1949 to 1953. This framework provides a natural synthesis of the spectral theory, the cohomology theory, and the classification of projection valued measures and thus of cohomology classes in terms of the Borel structure of the space on which the measure is defined (explicitly; the measure classes of this space). When certain regularity properties are assumed for this space (in particular when it is transitive) the classification arrived at is exhaustive.

We shall first introduce the imprimitivity theory and then consider the basic concepts from a more physical point of view. In (2.4.7) we develop the theory in a purely mathematical context. In (2.4.9) the elements of the decomposition theory of unitary representations are reviewed and the imprimitivity theory illuminated from the point of view of the Murray-von Neumann classification of operator rings. In Section 3.5 the possible physical applications of inequivalent representations of quantum fields are considered; this theory also makes use of the classification and it is helpful to see just why, for the spacetime symmetry groups, there is no such proliferation of representations. Of course the same body of mathematics forms the basis of the Stone-von Neumann theorem on the uniqueness of the Schrödinger rerpresentation of the CCR's.

The imprimitivity theory is concerned with the following things: a locally compact topological group G, a measure space X which carries an action of G, a Hilbert space $\mathcal{H}$, a projection valued measure P on X and a unitary representation of G by a set of operators U_g on $\mathcal{H}$. The problem is to relate $\mathcal{H}$ and U_g to G and X; the idea of such a measure space X (a **G-space** when we have a continuous action of G on X) has already played a crucial rôle in the foregoing; it has appeared as the dual to a commutative group. For semi-direct products X will be given in this way, as the dual to the translation sub-group, but for the general theory it is helpful to suppose we are given such a space *ab initio.* The basic idea is to generalize the notion of an *imprimitive* representation of a group, as developed by Frobenius, that is to represent the group action as a permutation on a set; we consider the set as the collection of subspaces of a separable Hilbert space and we consider the projection -valued measure on the G-space as a way of telling us how this permutation is to be effected. In order to do this we need to connect the projection valued measure (as a Boolean sub-algebra of the projection lattice of $\mathcal{H}$) to the representation U of the group (as a collection of unitary operators), so as to preserve a correspondence with the way the group acts on the G-space, and the way the projections are tied to the G-space (as a projection valued measure).

Thus motivated, we define a **system of imprimitivity** for G **based on** X as a pair (U_g, P_E) where U_g is a continuous unitary representation of G in a complex Hilbert space $\mathcal{H}$ and P_E is a projection-valued measure on X, such that $U_g P_E U_g^{-1} = P_{g.E}$ (g.E is the assumed action of G on X).

This is not quite the way the idea of a system of imprimitivity first arose; rather, if one considers the Weyl relationships in the form of Eq.(2), considering U as a unitary representation of the additive group N on $\mathbb{R}^n$ and V as a unitary representation of the dual $\hat{N}$, one may use the spectral theorem for the latter to conclude (writing $\mathbf{s} = n \in$

N, $t = \chi_t = \chi \in \hat{N}$):

$$(\phi, V_\chi \psi) = \int_N \chi(n)\, d(\phi, P_n \psi). \qquad (3)$$

Together with Eq.(2) one then has:

$$(\phi, \chi(n) V_\chi U_n \psi) = (\phi, U_n V_\chi \psi) = \int_N \chi(m) d(\phi, U_n P_m \psi).$$

But using Eq.(3) once again the LHS may be written:

$$\int_N \chi(m') d(\phi, P_{m'-n} U_n \psi)$$

from which follows:

$$U_n P_m = P_{m-n} U_n$$

that is, the imprimitivity relationship (the G-space here is N itself, with n.m = m-n). The Weyl relationship Eq.(2) is therefore equivalent to the requirement that $E \longrightarrow P_E$ should be a system of imprimitivity for U based on N. It was in this form that the imprimitivity relationship first appeared, in the context of the generalization of the Stone-von Neumann uniqueness theorem (Mackey [1949a]). In Mackey [1949b] much more general results were formulated[4].

As another example of this cluster of ideas, we see that a projection-valued measure on X sets up a correspondence between (bounded Borel) functions on X and bounded operators on $\mathcal{H}$: $(\phi, A_f \psi) = \int_X f(x)\, d(\phi, P_x \psi)$, with A_f self-adjoint if f is real. Given a system of imprimitivity, it then follows from the action of U on the projection valued measure that $U_g A_f U_g^{-1} = A_{g.f}$ where $g.f$ is the function $x \longrightarrow f(g^{-1}.x)$. Given enough topological structure on X one can reverse the construction and show that given such an algebraic homomorphism from $\mathcal{C}(X)$ into $\mathbf{B}(\mathcal{H})$ and given a representation U of G satisfying this equation then there is a unique projection valued measure P which gives rise to the algebraic homomorphism as above and U,P is a system of imprimitivity based on X.

There are other ways in which one can naturally associate a

[4] The central mathematical insights came in the relationship of measure theory to group cohomology; the relationships between measure theory and topology were already exploited in Stone's theorem: the map $g \longrightarrow U_g f$ from G to $\mathcal{H}$ is continuous if and only if the map $g \longrightarrow (f, U_g f)$ is measureable on G for all $f \in \mathcal{H}$.

system of imprimitivity with sub-algebras of $\mathbf{B}(\mathcal{H})$, and one can even define an *abstract* C^*-algebra from a pair G,X (with X a G-space)[5]. Alternatively when G is a spacetime group, if one can think of X as configuration space, one can consider the projection-valued measure P_E abstractly, as a Boolean algebra of properties[6] corresponding to statements of the form: "the system Ω is localized in E". The imprimitivity relationship

$$U_g P_E U_g^{-1} = P_{g.E}$$

clearly expresses the idea that such properties will change under the action of the group in the intuitively obvious way, corresponding to the idea of the locality of Ω in X. This is particularly transparent when Ω is considered an (elementary) particle; proceeding in this way one will be assured of the existence of operators which describe the position of a particle, because we can define such operators from the P_E's via the spectral theorem.

Whilst the spacetime groups have, by definition, a natural action on spacetime, it is not clear that we may consider configuration space as a G-space; obviously with spacetime as base one will obtain a notion of localization which does not fit with our intuitions of a particle localized in *space* and persisting in time. In this situation one naturally looks to a sub-group of the relativity groups which does not include the time-translations (or the velocity boosts); defining position operators in this way one must then face up to the question of how this representation is to be related to the entire group. There, in a nutshell, is the heart of the difficulty of making sense of the notion of locality in RQM. This is pursued at length in Part 3,

[5] See, for example, Segal [1951], who defined two kinds of abstract algebras in this way. One of these will be of some relevance to Section 3.5, particularily (3.5.3), which is concerned with Haag's theorem.

[6] In the sense of Section 2.2. This approach is particularily natural if one begins from the Mackey model, and has been used (in non-relativistic theory!) to obtain a Hilbert space representation theory for an abstract lattice without some of the more technical assumptions of the Piron theorem; see Gudder [1973],[1977].

particularly (3.2.2), (3.2.8), (3.4.7), (3.4.8).

As indicated in the introduction, more generally we consider spacetime as secondary to the notion of a physical system Ω described by a fundamental model. We assume rather that given such a system we introduce spacetime in terms of a group of transformations on the fundamental model which preserve its algebraic structure, *i.e.* we require that G acts *automorphically* on the fundamental model. If we view the imprimitivity relationship as an expression of the *covariance* of the description of Ω under G, the former is an expression of the *invariance* of the structure of Ω under G. The history of group theory bears ample evidence to the subtle interblending of these two ideas; from its mathematical origins in permutation groups (the simplest finite *closed* algebras), and their applications in invariant theory and the Gal-*o*is theory, to its physical origins in geometry and the idea of a transformation of a coordinate system. The idea of a G-space is fundamental to the notion of covariance.

In QFT, whilst these remarks hold true when we consider the fields as a single system Ω, we shall also consider the physical system as defined (or "internally structured") according to spacetime intuitions. We consider this to *define what we mean* by a field theory. This builds a new action of the group into the theory (covariance of the fields), which acting on the system as a whole is also automorphic. In the latter context the sequel is applied just as for the 1-particle system; however the imprimitivity theory will not here or subsequently be related to the covariant action of the group on the quantum fields. Concerning the notion of localization, one has no problem with the concept of localization in space *and* time - if only one could make sense of the idea that the field is localized at all. That is, when Ω is a quantum field, it is clearly meaningless to think of Ω as localized.

This discussion is continued in the next section. Let us now postpone these questions and study the imprimitivity theory

in its own right. The following results were all obtained by Mackey in [1949b] and [1952]; for proofs of theorems we refer to Vardarajan [1970], a self-contained treatment. For a concise summary, see Mackey [1963b].

2.4.7. Systems of imprimitivity and classification theory

The simplest example of a system of imprimitivity is the following; let $\mathcal{H} = L^2(X,\mu)$ with μ a σ-finite invariant measure on X (so that $X = \bigcup_i E_i$, with each E_i Borel and $\mu(E_i)$ finite for all i, and $\mu(E) = \mu(g.E)$ for all Borel E, $g \in G$), and define:

$P_E f = \chi_E f$,

$(U_g f)(x) = f(g^{-1}.x)$

(here χ_E is the characteristic function of E). It is simple to verify that the pair U,P has all the requisite properties. A slightly more complicated but physically more relevant system relaxes the assumption of invariance of the measure and permits the use of vector-valued (spinor-valued) functions on X. Given a **quasi-invariant** measure (that is, mututally absolutely continuous with an invariant measure, or having null sets invariant under G), we have only to correct for the change in measure under the action of G. This may be done using the Radon-Nikodym derivative of μ with respect to its translation $g^{-1}[\mu]$ under the inverse action of G, which we denote r_g. This is itself a Borel function on X which obeys the identity:

$$r_{g_1 g_2}(x) = r_{g_1}(g_2.x)\, r_{g_2}(x) \quad \text{or} \quad r_{g_1 g_2} = g_2^{-1}[r_{g_1}]\, r_{g_2} \qquad (4)$$

Consider now the operator U_g defined by

$U_g f = (g[r_g])^{1/2} g[f]$, or $(U_g f)(x) = (r_g(g^{-1}.x))^{1/2} f(g^{-1}x)$.

Since $U_{g_1} U_{g_2} f = U_{g_1} (g_2[r_{g_2}])^{1/2} g_2[f] =$

$(g_1[r_{g_1}]\, g_1[g_2[r_{g_2}]])^{1/2} g_1[g_2[f]]$ and $g_1[g_2[f]] = g_1 g_2[f]$

we will have that $U_{g_1} U_{g_2} f = U_{g_1 g_2} f$ if only $g_1[r_{g_1}]\, g_1 g_2[r_{g_2}] = g_1 g_2[r_{g_1 g_2}]$; but that is exactly what Eq.(4) ensures. The square root is needed in order to make U an isometry; $\|U_g f\|$

$= \int_X g[r_g]\overline{g[f]}g[f]d\mu = \int_X r_g(g^{-1}.x)\overline{f(g^{-1}.x)}f(g^{-1}.x)d\mu(x) =$
$\int_X r_g(x)\overline{f(x)}f(x)d\mu(g.x)$ and by definition of the Radon-Nikodym derivative $\int f(x)\, r_g(x)d\mu(g.x) = \int f(x)\, d\mu(x)$.

To generalize the foregoing to vector or spinor valued functions on X is much easier; we define $\mathcal{H}$ as $L^2(X,\mathcal{K},\mu)$ with $\mathcal{K}$ a (separable) Hilbert space with inner product $<.,.>$ (the spin space); $\mathcal{H}$ is the Hilbert space of $\mathcal{K}$-valued functions on X, with inner product $(f,f')= \int <f(x),f'(x)>d\mu(x)$. The system of imprimitivity is as before.

These systems are called **Koopman systems**. We have already studied them in a slightly different context, that is as Hilbert space realizations of the algebra of classical observables; Koopman also showed that the classical evolution is represented as a unitary representation of the additive group of the reals. We now see that this too is a special case of a more general theory. But this generalization, the Koopman systems as defined above, is still not quite general enough to provide an exhaustive classification. The group might also act in the separable Hilbert space $\mathcal{K}$ (as our terminology suggests). The imprimitivity theory now gives us the relationship between the quasi-invariant measure μ occurring in this representation (more properly its measure class, *i.e.* the set of all measures with respect to which it is absolutely continuous), the equivalence class of the projection valued measure, and the cohomology theory of the group G. I shall only summarize the most significant results. To do this, we need some definitions. First, the identities Eq.(4) are an example of the **co-cycle identities.** They play a pivotal rôle in the cohomology theory of groups, a rapidly developing field in the late 1940's, and to which Mackey himself contributed. Let ϕ be a function on GxX with values in a Borel group M. It is called a **strict (G,X,M) cocycle** if the condition

$$\phi(g_1g_2,x) = \phi(g_1,g_2.x)\phi(g_2,x),\ \phi(e,x) = 1 \qquad (5)$$

is satisfied. A similar definition for almost all triples g_1,g_2,x relative to some (product) measure is also useful;

it is then a (G,X,M) cocycle relative to the measure class of this measure (and likewise in the following definitions). Two strict cocycles ϕ_1, ϕ_2 are called **strictly cohomologous** if there is a map $d: X \longrightarrow M$ such that $\phi_2(g,x) = d(g.x)\phi_1(g,x)d(x)^{-1}$. This relationship is an equivalence relationship and the equivalence classes of cocycles thus determined are called the **strict cohomology classes** of G. Fixing a measure class $\mathcal{C}$ in X and repeating the foregoing definitions for almost all x,g, leads to the cohomology relative to the measure class $\mathcal{C}$.

We shall not develop the cohomology theory here but from the similarity of Eq.(4) and Eq.(5) it is apparent that we can generalize the Koopman systems and connect the spin space of the representation with the group action if the Borel group M acts on $\mathcal{K}$, simply by including a cocycle along with the Radon-Nikodym derivative. Eq.(5) then guarantees that the representation U_g is a homomorphism and if we make M the unitary group on $\mathcal{K}$ it is also an isometry. We thus obtain a correspondence between unitary representations and a strict cohomology class, or a cohomology class relative to a measure class $\mathcal{C}$ on X. In fact Mackey proved the following theorem.

Theorem 2.4.1.

For each quasi-invariant measure μ in the measure class $\mathcal{C}$ and for each (G,X,M)-cocycle ϕ relative to $\mathcal{C}$ there is a unique system of imprimitivity P,U, with Hilbert space $\mathcal{H} = L^2(X,\mathcal{K},\mu)$ such that:

$P_E f = \chi_E f,$

$(U_g f)(x) = (r_g(g^{-1}.x))^{-1/2}\phi(g,g^{-1}.x)f(g^{-1}.x).$

Moreover, the (unitary) equivalence class of U,P is uniquely defined by the measure class $\mathcal{C}$ and the cohomology class of ϕ. (Proof: Varadarajan [1970 Th.9.7].)

What is more remarkable is that given an arbitrary system of imprimitivity P,U, if P can be brought into the form of Theorem 2.4.1 by a unitary transformation then there must exist a measure class $\mathcal{C}$ and a (G,X,M)-cocycle relative to $\mathcal{C}$

such that U is also of this form. Such a P, *i.e.* any projection valued measure P with values on $\mathcal{H}$ such that $P = WP'_0W^{-1}$, $P'_Ef = \chi_Ef$ for $f \in L^2(X,\mathcal{K},\mu)$, with W a unitary map from this L^2 space to $\mathcal{H}$, is called **homogeneous.** The measure class which occurs in this canonical representation is uniquely determined by P; it is the class of sets E such that $P_E=0$. When P is homogeneous it is easy to see that its measure class is the same as that of the measure which is used in the canonical construction of $\mathcal{H}$.

For our purposes the assumption that P is homogeneous is not too burdensome. If U,P is an *irreducible* system of imprimitivity then P is homogeneous. The conclusion also follows from the weaker premise that the measure class of P is **ergodic**; that is when there is no measure μ in this class such that there exists a μ-measurable subset of X invariant under G which is not μ-null or the complement of a μ-null set. It is **transitive** if there exists $x_0 \in X$ such that $G.x_0$ differs from X only on $\mathcal{C}$-null sets. Every transitive measure class is ergodic; if the measure class of P is transitive, then we say U,P is a transitive system of imprimitivity and P is then homogeneous. In this case the measure classes of P are in one-one correspondence with the cohomology classes of G.

The weakening of the ideas of ergodicity and transitivity used here is actually crucial to the rigorous application of these ideas to statistical mechanics, in particular in the context in which they were originally developed by Boltzmann: the action of the evolution in time on the energy hypersurfaces in phase space. Each energy hypersurface is a G-space, and the group G is the additive group of the reals. It was actually in this context that von Neumann, following discusions with Koopman, perceived that the Koopman system as a Hilbert space model of classical mechanics suggested the possibility of using operator methods to prove the equality of space and time averages in statistical mechanics. The crucial step in his proof that this was possible (the mean ergodic theorem) was the slight weakening of the concepts of ergodicity and transitivity used by

Boltzmann.

The foregoing may be summarized as follows:

Theorem 2.4.2

Let U,P be a system of imprimitivity with P transitive or ergodic. Let $\mathcal{C}$ be its measure class. Then there is an integer n ($1 \leq n \leq \infty$), and a cocycle ϕ relative to $\mathcal{C}$, and a measure μ in $\mathcal{C}$, such that U,P is equivalent to the system U'P' acting in $\mathcal{H} = L^2(X, \mathcal{K}, \mu)$ defined by $P'_E f = \chi_E f$, $(U'_g f)(x) = (r_g(g^{-1}.x))^{1/2} \phi(g, g^{-1}.x) f(g^{-1}.x)$, where n is the dimensionality of $\mathcal{K}$. (Proof: Varadarajan [1970 Th. 9.7, 9.11]).

2.4.8 Decomposition Theory.

The imprimitivity theory classifies systems of imprimitivity, not unitary representations. Given an arbitrary representation of a group G we do not know how to determine its measure class and cohomology class and we do not even know if there exists a system of imprimitivity to which it is unitarily equivalent.

For certain types of groups, it is a remarkable fact that we obtain *all* unitary representations as systems of imprimitivity. The inhomogeneous Lorentz group is one of these groups. This result is based on the analysis of the possible operator algebras that can arise on a separable Hilbert space.

Most of the ideas following are contained in Murray and von Neumann [1936] and von Neumann [1949]; their systematization in application to the theory of l.c. groups is due to Mackey [1953], [1957], although some preliminary work along these lines had been done by Mautner [1950], Godement [1951], and Segal [1951]. We follow Mackey [1963a]. It is clear that we are essentially going to use the Murray-von Neumann classification scheme for von Neumann algebras, and that the same ideas apply to the classification of C^*-algebras. Indeed the decomposition theory of operator

algebras and l.c. non-compact groups were developed in parallel; amongst the motivations listed for the study of factor representations of operator algebras, a really crucial step from which the entire theory of operator rings proceeded, Murray and von Neumann ([1936]) included the consideration of factor representations of the ring generated by a unitary representation of a group on a *finite* dimensional Hilbert space, from the point of view of the classical theory of unitary representations (the Burnside-Frobenius-Schur theorems).

The most important object in the decomposition theory, given a unitary representation L of a group on a Hilbert space (which we suppose to be fixed throughout), is the commutant of all the elements of this set of unitary operators, called the **commuting ring** of the representation, denote $\mathcal{R}(L,L)$. It is a special case of the set of **intertwining operators** for any two unitary representations U,V, denoted $\mathcal{R}(U,V)$: the set of all bounded maps T: $\mathcal{H}_U \longrightarrow \mathcal{H}_V$ such that $TU_g = V_g T$. Clearly if $\mathcal{R}(U,V)$ contains a unitary mapping then the representations U and V are unitarily equivalent (which we shall at times denote by $U \simeq V$).

The significance of the commuting ring of a (continuous unitary) representation L is easy to see. Suppose $\mathcal{H}^0$ is an invariant subspace of $\mathcal{H}$ under L and L^0 denotes the restriction of L to $\mathcal{H}^0$; then clearly $g \longrightarrow L^0_g$ is a (continuous unitary) representation of G on $\mathcal{H}^0$ If we define P_0 as the (unique) projection onto $\mathcal{H}^0$ we call L^0 the **subrepresentation of L corresponding to P_0**. If the range of P_0 is invariant under G, then P_0 must lie in $\mathcal{R}(L.L)$, so the subrepresentations of L are in one-one correspondence with the projections in $\mathcal{R}(L,L)$. For a general $P \in \mathcal{R}(L,L)$ denote its corresponding subrepresentation L^P; clearly $L \simeq L^P \oplus L^{1-P}$ (where $P^{\perp} = 1-P$) and L is irreducible if and only if $\mathcal{R}(L,L)$ contains no non-trivial projection (in which case it follows from the spectral theorem that it contains only multiples of the identity).

An arbitrary (non-zero) element T in $\mathcal{R}(L,M)$ implies an

equivalence of *some* subrepresentation of L with *some* subrepresentation of M, as can be seen as follows. If $\mathcal{K}_T$ is the kernal of T and $\mathcal{R}_T$ the closure of its range it follows that these spaces are invariant under L,M respectively and by Schur's lemma that the subrepresentation of L corresponding to $1-P_{\mathcal{K}_T}$ is unitarily equivalent to the subrepresentation of M corresponding to $P_{\mathcal{R}_T}$. Thus motivated, if $\mathcal{R}(L,M)$ contains only the zero, L and M are called **disjoint**. This notion will be of considerable importance to the measurement theory discussed in Section 3.5 for C^*-algebras. It also leads to a new notion of equivalence: if no non-zero subrepresentation of L is disjoint from M and no non-zero subrepresentation of M is disjoint from L then L and M are called **quasi-equivalent.** For example, U is always quasi-equivalent to U⊕U (though if irreducible clearly not unitarily equivalent). In fact the notion of quasi-equivalence arose in classical theory in order to associate representations in which the same irreducible sub-representations occur, but perhaps with different multiplicities. Murray and von Neumann developed a generalized theory in which continuous decompositions and continuous multiplicities are permitted and from this theory two definitions of quasi-equivalence appeared; the one, due to Mackey [1953] as above, and the other in the representation theory of C^*-algebras, most particularly in connection with the relationship of two representations π_1, π_2, of a C^*-algebra $\mathcal{A}$ with the associated von Neumann algebras $\pi_i(\mathcal{A})''$. We say that the π_i are quasi-equivalent if there exists a * isomorphism α from $\pi_1(\mathcal{A})''$ to $\pi_2(\mathcal{A})''$ such that $\alpha(\pi_1(\mathcal{A}))=\pi_2(\mathcal{A})$. The two definitions were shown to coincide by Dixmier [1964 Prop.5.3.1].

The centre $\mathcal{C}(L)$ of the commuting ring $\mathcal{R}(L,L)$ of a representation L is of particular importance; this is evident from the fact that for a projection P in $\mathcal{C}(L)$ the corresponding subrepresentations L^P, L^{1-P}, are disjoint. The converse is also true and it follows that L cannot be decomposed into the direct sum of disjoint parts when $\mathcal{C}(L)$ is trivial. In that case the representation L is called

primary. There may still exist non-trivial projectors in $\mathcal{R}(L,L)$, and a primary representation may be further decomposed into (non-disjoint) subrepresentations. This is obviously impossible when $\mathcal{R}(L,L)$ is commutative (in which case it coincides with $\mathcal{C}(L)$); L can therefore be decomposed into the direct sum of (disjoint irreducibles and is called **multiplicity free.** Clearly if $L \simeq L^1 \oplus L^2 \oplus \ldots$ and each L^j is irreducible then L is multiplicity free if and only if $L^i \simeq L^j$ implies i=j, and L is primary if and only if $L^i \simeq L^j$ for all i,j.

To further explore the various possibilities that may arise we note that the commuting ring of a representation is always von Neumann and that the Murray - von Neumann classification is immediately applicable. In particular a representation L is primary (as defined above) if and only if its commuting ring is a factor in the sense of Murray and von Neumann (that is, a von Neumann algebra $\mathcal{A}$ is a factor if $\mathcal{C}(\mathcal{A}) = \mathcal{A} \cap \mathcal{A}'$ is trivial). To understand the classification scheme it is necessary to review the generalized notion of trace; we define a **trace** on a von Neumann algebra $\mathcal{A}$ as an element f in $\mathcal{G}$ which has the property: $\langle f; AA^* \rangle = \langle f; A^*A \rangle$ for all $A \in \mathcal{A}$. We say a trace f is **faithful** if for A positive, $\langle f; A \rangle = 0 \Rightarrow A = 0$, **semifinite** if $\langle f; A \rangle = \sup_{B \leq A} \langle f; B \rangle$ for all positive $A, B \in \mathcal{A}$, **finite** if $\langle f; A \rangle$ is finite for all $A \in \mathcal{A}$. We then say that $\mathcal{A}$ is **semifinite** (**finite**), if there exists a semifinite (finite) trace on $\mathcal{A}$ which is also faithful and normal; it is **properly (purely) infinite** whenever f=0 is the only finite (semifinite) normal trace on $\mathcal{A}$. If $\mathcal{A}$ is isomorphic to a von Neumann algebra $\mathcal{A}'$ with $\mathcal{R}(\mathcal{A}', \mathcal{A}')$ abelian, it is said to be **discrete**. If there is no non-zero projector P in $\mathcal{C}(\mathcal{A})$ such that $\mathcal{A}_P$ is discrete then $\mathcal{A}$ is called **continuous.**

We can now state the Murray von Neumann classification:

Theorem 2.4.3.

Let $\mathcal{A}$ be an arbitrary von Neumann algebra. Then there exists a partition of the identity by five projectors $E_I(n)$, $E_I(\infty)$, $E_{II}(1)$, $E_{II}(\infty)$, E_{III} in $\mathcal{C}(\mathcal{A})$ such that the corresponding subrepresentations of $\mathcal{A}$ are, respectively, discrete and finite (type I_n), discrete and properly infinite (type I_∞), continuous and finite (type II_1), continuous and properly infinite (type II_∞), and purely infinite (type III). If $\mathcal{C}(\mathcal{A})$ is trivial $\mathcal{A}$ is necesssarily one of these types.
(Proof: Murray and von Neumann [1936 Th.VIII], Dixmier [1957 III.2.7, prop. 14]).

For orientation, a von Neumann algebra on an n-dimensional Hilbert space is always of type I_n; a commuting von Neumann algebra is always finite, hence of type I_n or II_1; $B(\mathcal{H})$ with $\mathcal{H}$ infinite dimensional is of type I_∞. What also distinguishes type I representations, from the point of view of the states ω on $\mathcal{A}$ and in particular the pure states $\omega^P \subseteq \omega$, is that if π is a primary representation of a C^*-algebra then the vector states of π cannot be pure if π is of type II or III. The pure states of an algebra can only be realized as vector states in type I representations.

Following Mackey [1953], we say that a primary representation L of a group is **of type** I_n, I_∞, II_1, II_∞, III if and only if its commuting ring is of **type** I_n, I_∞, II_1, II_∞, III. The classification of primary representations of a unitary group is therefore identical to the classification of factors, and is thus given above. It can be shown that a primary representation of L is of type I if and only if it has an irreducible subrepresentation, and also if and only if it is unitarily equivalent to a representation of the form $nM = M \oplus M \oplus \ldots \oplus M$ (n times), with M irreducible; n is called the **multiplicity** of the representation. This follows from the fact that quasi-equivalent representations are of the same type. It is then obvious that given a representation $L \simeq L^1 \oplus L^2 \oplus \ldots L^i \ldots$ with each L^i primary and disjoint, then each L^i is:

Type I iff L is quasi-equivalent to a multiplicity free

representation.

Type II iff L is quasi - equivalent to a finite representation and every representation quasi-equivalent to L is of the form M⊕M for some M.

Type III iff every representation quasi-equivalent to L is equivalent to L.

These criteria clearly make sense even when L is not written as a direct sum of disjoint factors; in particular we shall say that for arbitrary L, L is of type I if and only if it is quasi-equivalent to a multiplicity free representation. This is equivalent to the definition that L is of type I if and only if it can be written as a direct sum $L = L^1 \oplus L^2 \oplus \ldots \oplus L^j \oplus \ldots$ with each L^j multiplicity free. One can now reduce the problem of determining all type I representations to the problem of determining all multiplicity free representations by means of the following theorem (Mackey [1953 Th.1.2]):

Theorem 2.4.4.

Let L be any representation of type I. Then there exists a sequence $L^\infty, L^1, L^2, \ldots$ of disjoint multiplicity free representations such that L is unitarily equivalent to the representation $\infty L^\infty \oplus L^1 \oplus 2L^2 \oplus 3L^3 \oplus \ldots$. The L^j are unique up to unitary equivalence, but in each nL^n some of the summands may be missing.

We shall now consider the generalized direct sum of von Neumann [1949], as adapted by Mautner [1950] and Mackey [1957].

Let S be a standard Borel space and μ a finite measure on S and $\mathfrak{h}$ a Hilbert space. We suppose for each $s \in S$ there exists a unitary representation L^s of the group G which is Borel on $S \times G$ in the sense that $\langle L^s_g \phi, \psi \rangle$ is Borel for all $\phi, \psi \in \mathfrak{h}$, with $\langle .,. \rangle$ the inner product in $\mathfrak{h}$. We define a new representation L on the Hilbert space $\mathcal{H} = L^2(S, \mathfrak{h}, \mu)$ with inner product $(f,g) = \int_S \langle f(s), g(s) \rangle d\mu(s)$ as follows; for each $f \in \mathcal{H}$ and $g \in G$ define $(L_g f)(s) = L^s_g(f(s))$. Following Mackey, we call L the **direct integral** of L^s with respect to

S and denote it $L = \int_S L^s d\mu$. This construction will be slightly generalized in section (3.3.6) to obtain the covariant representations defined by the wave equations, to include the situation when for each $s \in S$ we have a different Hilbert space $\mathfrak{h}_s$ with inner product $\langle .,. \rangle_s$ (that is, the L^s act in different spaces). The regularity condition must be maintained, however, and the dimension of $\mathfrak{h}_s$ must be a measurable function of s.

The connection with projection valued measures on S becomes clear when we observe that if P_E is the projection in $\mathcal{H}$ for each Borel subset of S such that $P_E f = \chi_E f$ (with χ_E the characteristic function of E) then P_E is contained in the commuting ring of the direct integral representation L on $\mathcal{H}$. Mackey [1953] showed that if M is any unitary representation of G and P^M a projection valued measure on a space S with P^M_E in $\mathcal{R}(M,M)$ for all Borel $E \subseteq S$ then there exists an assignment $s \longrightarrow L^s$ of a unitary representation to each s in S and a finite measure μ in S such that $L = \int_S L^s d\mu(s)$ exists and is equivalent to M. In particular the null sets of μ are those E for which P^M_E is zero and within unitary equivalence each L^s is uniquely determined for almost all s. In this way any unitary representation L of G together with a projection valued measure P such that $P_E \in \mathcal{R}(L,L)$ is the same thing as a representation L together with a particular realization of L as a direct integral.

One can in fact eliminate the space S from the foregoing and define a correspondence between representations of G and Boolean algebras of projectors in $\mathcal{R}(L,L)$. This is so because given such an algebra $\mathcal{F}$ there must exist a projection valued measure P and a standard Borel space S which has $\mathcal{F}$ as its range, with S essentially unique. In particular one can show that given L, $\mathcal{F}$, then almost every L^s is irreducible if and only if $\mathcal{F}$ is maximal in $\mathcal{R}(L,L)$. Therefore from Zorn's lemma *every* unitary representation L has a direct integral decomposition into irreducible parts. However, this decomposition is in general far from unique; for example, it is possible for $L \simeq \int_S L^s d\mu(s) \simeq \int_{S'} L^{s'} d\nu(s')$ such that no L^s is unitarily equivalent to any $L^{s'}$.

We obtain a canonical decomposition into disjoint primary parts by taking $\mathcal{F}$ to be all the projections in $\mathcal{C}(L)$; this is called the **central decomposition** of L. When L is multiplicity free $\mathcal{C}(L)$ coincides with $\mathcal{R}(L,L)$ so $\mathcal{F}$ is in this case automatically maximal; therefore a multiplicity free representation L has a canonical (central) decomposition into a direct integral of irreducible representations. This decomposition is unique.

We shall denote by $\mathfrak{G}$ the set of all unitary equivalence classes of irreducible representations of a group G; if L is multiplicity free let $L = \int_S L^s d\mu(s)$ be the central decomposition. It then defines a map $S \longrightarrow \mathfrak{G}$ given by $s \longrightarrow L^s$ and a measure m in $\mathfrak{G}$ such that $\mu(E) = m(L^E)$ (where L^E is the set of all L^s with $s \in E$), and the measure class C_m of m determines L to withiin unitary equivalence. The most important step in Mackey [1957] was to define a natural Borel structure in $\mathfrak{G}$; he showed that every C_m is a Borel measure class, although not every Borel measure class determines a multiplicity free representation. However $\mathfrak{G}$ is a standard Borel space if and only if every representation of G is of type I, and if $\mathfrak{G}$ is a standard Borel space then every Borel measure class in $\mathfrak{G}$ arises from a multiplicity free representation[7]. We have already seen that type I representations can be given as the direct sum of multiplicity free representations; we now see that in this case the multiplicity free representations are in one-one correspondence with the Borel measure classes in $\mathfrak{G}$, or equivalently with the Borel measure classes in a standard Borel space S. It is clear that an irreducible representation corresponds to a measure in $\mathfrak{G}$ concentrated on a point, and that an arbitrary type one representation is a direct sum of multiples of multiplicity free representations, each of which is a direct sum of

[7]This is essentially a generalization of the classical Hahn-Hellinger theory for self-adjoint operators in Hilbert space. In [1957] Mackey required that $\mathfrak{G}$ be standard and have only type 1 representations as independent assumptions; the equivalence of these assumptions stated above was first proved by Glimm [1961].

irreducibles if and only if the corresponding measure class is concentrated on a countable set of points in $\mathfrak{S}$. Otherwise it is uniquely given as a direct integral of irreducible representations.

Amongst the groups which are known to have only type I representations are the compact groups, the commutative groups, and the connected semisimple Lie groups (this result is due to Harish-Chandra [1953]). The homogeneous Lorentz group in particular has only type I representations.

We see the general relationship between projection-valued measures on a Borel space and the classification theory; but $\mathfrak{S}$ and m are given abstractly. It is here that a second enormous simplification is possible for the inhomogeneous Lorentz group: it is the semi-product of the Lorentz group L with an abelian normal group T (the spacetime translations). For semi-products of this form there is a canonical G-space, namely the dual space $\hat{T}$; it is a fundamental result of Mackey [1949b] that for semi-product groups (the equivalence class of) every irreducible representation is defined by a Borel measure class in $\hat{T}$. Essentially there is a correspondence between the space of all irreducible representations of T (denote $\mathfrak{X}$) and the space $\mathfrak{S}$, and therefore also between C_m in $\mathfrak{S}$ and the measure classes of $\mathfrak{X}$; but $\mathfrak{X}$ can be identified with $\hat{T}$. In this way the decomposition theory ultimately reduces to a study of the measure classes of *momentum space*.

Obviously this is also true for the most general representation of a *commutative* group G. Commutative groups are also type 1; in view of the interpretation of the Weyl relationships as the condition that there exists a system of imprimitivity for N based on N, where N is the additive group on $\mathbb{R}^n$, it is clear that the classification theory for the representation of the Weyl relationships is essentially trivial: the most general representation is the direct sum or direct integral of irreducible representations.

We shall return to this decomposition theory in (3.1.3) in

our review of the Wigner classification, where we consider semi-direct products in more detail. There we shall also need the powerful tool originally developed by Frobenius and generalized by Mackey [1949b], [1951]: the *inducing construction.*

2.4.9. The inducing construction.

From now on we restrict attention to *transitive* systems. The justification of this assumption will ((3.1.3)) derive from assumptions about the smoothness of the orbit structure; for an elementary particle that is related to continuity assumptions on the dual to space-time, therefore to space-time itself. For such systems the projection-valued measure is homogeneous and there is a one-one correspondence between equivalence classes of systems of imprimitivity, transitive measure classes on X, and cohomology classes of G. Further we know that a system of imprimitivity U,P, can be written in the canonical form of Theorem 2.4.2. We now consider the classification of the *irreducible* representations of G. For transitive systems this can be elegantly formulated in terms of the *inducing construction*, which we have met in the finite dimensional case already.

The fundamental idea is that for each point x_0 of X the **stability group** G_0 (g in G such that $g.x_0=x_0$) may have a simple representation theory which can be used to classify the representation of the group G. If one considers the canonical form of U_g on $L^2(X,\mathcal{K},\mu)$, that is:

$$U'_g f)(x) = (r_g(g^{-1}.x))^{1/2}\phi(g,g^{-1}.x)f(g^{-1}.x),$$

it is apparent that the restriction of the (G,X,M)-cocycle to the stability group at x_0 gives a representation of G_0 on $\mathcal{K}$; the assumption of transitivity plays a crucial rôle here, in allowing one to conclude that this representation must be independent of the point x_0, but depends only on the orbit of the point x_0 under the action of G. For each point x_0 we therefore obtain a homomorphism of G_0 into the unitary group M on $\mathcal{K}$, and for each orbit an

equivalence class of such homomorphisms (all the representations of G_x for $x \in G_0$ are unitarily equivalent). The fundamental result obtained by Mackey is that the equivalence classes of homomorphisms of the stability group at x_0 into M are in one-one correspondence with the (G,X,M)-cohomology classes of G. This means that given any representation L of G_0 on $\mathcal{H}$ one can determine a strict cocycle and from that construct the corresponding representation of G; it is called the **induced representation** and is denoted U^L. This was the central result in his theory of induced representations; it means, amongst other things, that the representation U^L is irreducible whenever the representation L is. In physical applications the stability group is particularly simple (for massive particles the homogeneous part is *compact*), so one can reduce the classification of the representations of G to that of much simpler groups.

I have throughout assumed that we have a strict cocycle, and correspondingly that X is transitive and not just that we have a transitive measure class. Actually this is no restriction; given a transitive measure class and a point x_0 we just take the orbit $X' = G.x_0$ as our G-space (so that X' *is* then transitive even if X is not) and the (G,X,M)-cohomology classes relative to $\mathcal{C}$ become strict (G,X',M)-cohomology classes. In this way the measure class and with it the cohomology class of the representation can be uniquely specified by a choice of orbit $G.x_0$ and the representation in M of the stability group at x_0. At the same time the system of imprimitivity is determined, up to equivalence, to be of the form given in Theorem 2.4.2 with a strict cocycle in the strict (G,X',M) cohomology class corresponding to the representation L. We call this the system *induced* by $\mathcal{C}$ and L. In summary we have the fundamental theorem (Varadarajan [1970 Th.9.12, Cor.9.13]):

Theorem 2.4.5.

For any transitive system of imprimitivity U,P, where the measure class of P is associated with the orbit $X' = Gx_0$, then there exists an integer n and a representation L of the stability group G_0 at x_0 on $\mathcal{K}$ (of dimension n) such that U,P is equivalent to the system induced by $\mathcal{C}$ and L. This equivalence class of systems of imprimitivity depends only on $\mathcal{C}$ and the equivalence class of L. For fixed $\mathcal{C}$ the set of all equivalence classes of systems of imprimitivity is in one-one correspondence with the $(G_0, \mathcal{K}, M)$-cohomology classes relative to $\mathcal{C}$. The induced system is irreducible if and only if L is irreducible.

As we have already indicated, for the inhomogeneous Lorentz group the dual to the translation subgroup is a canonical G-space; the orbits are the mass hyperbola and for positive mass the stability sub-group at any point on an orbit is *compact*. The representation theory then reduces to the theory for compact groups, and is completely known.

2.4.10. The uniqueness theorem.

It can easily be seen from the material of (3.1.3) that for the translation group N on $\mathbb{R}^n$ the most general possible irreducible system of imprimitivity U,P, for N based on N is transitive. We may therefore apply theorem 2.4.5. The stability subgroup for any point $x \in N$ is trivial, that is the identity, and since this has a unique irreducible representation there is a unique irreducible representation of N. Up to unitary equivalence, there is only the representation of Eq.(1); the Schödinger representation of the Weyl relationships is essentially unique.

Theorem 2.4.5, and all the theorems that we have discussed, apply only to *locally compact* groups. N is of course locally compact when it is the translations on $\mathbb{R}^n$. If we consider the infinite dimensional analogue, which is essentially the situation when we consider the Weyl algebra for a quantum field theory, N is no longer locally compact; the uniqueness theorem fails.

2.5. Quantum field theory

> The possibility of describing the world by means of Newtonian mechanics tells us nothing about the world: but what does tell us something about it is the precise *way* in which it is possible to describe it by these means. We are also told something about the world by the fact that it can be described more simply with one system of mechanics than with another.
>
> L. Wittgenstein

2.5.1. Overview.

In Section 3.1 we shall study the representations just discussed, and in 3.3 their relationship to the covariant wave equations. In Section 3.2 the Mackey theory is also applied in connection with the definition of position operators. This section is devoted primarily to another strategy altogether, defining the group action on an abstract algebra in a different, but very simple manner. We shall still obtain a unitary representation of the Lorentz group, so that there will still exist a projection valued measure (based on $\hat{A}$) corresponding to the properties "the system Ω lies in a subset of $\mathbb{R}^4$" (that is, "the field has total energy - momentum in a subset of $\mathbb{R}^4$"), but we shall not define the algebra in terms of these observables, nor the group action on $\mathcal{A}$ in these terms. We *shall* demand that the measure class of this measure is concentrated on *positive* values of p_o, but the system will be highly reducible (in a group theoretic sense); this measure shall not be concentrated on any orbit. This type of connection between a physical system and a space-time group is so fundamental that we may say that it *defines* the idea of a field.

This approach is elaborated in (2.5.3) below; (2.5.2) gives the necessary background on the automorphisms of C^*-algebras. With (2.5.5) we complete our programme of a realist interpretation of the *general* structure of a quantum field theory. Thereafter our focus is much more specific: we shall consider the Fock representation of a class of quantum field theories, which are the quantization of linear classical fields. That is, we assume we are given a Weyl or Clifford algebra *ab initio*, those which are associated with a linear classical phase space; in obtaining the Fock space representations for these algebras, we are essentially quantizing the associated classical theories. As we have mentioned, in the finite dimensional case, and particularly for the equal-time CCR's and ACR's, this can be looked at in a purely group theoretic way, but we shall not attempt to pursue this interpretation in the infinite dimensional case. Rather, we shall emphasise the connection with the general theory of quantization.

Part 3 is mainly concerned with the particle interpretation of the standard free theories, in which no deeper aspects of the general theory are used. But at the end we shall discuss more general problems of interpretation, and in particular the measurement problem, which are connected with the existence of inequivalent representations for field theories (Section 3.5).

To begin, we consider the general idea of the invariance of an algebraic system, which we shall consider roughly equivalent to the *form invariance* of the fundamental laws governing the behaviour of the system.

2.5.2. Group theory and algebraic invariance.

Consider a Segal model $\mathfrak{U},\mathfrak{G}$. An **automorphism** of $\mathfrak{U}$ is a linear map α on $\mathfrak{U}$ onto itself which also preserves the "quasimultiplication", *i.e.* $\alpha(A+\lambda B) = \alpha(A)+\lambda\alpha(B)$, $\alpha(A\circ B) = \alpha(A)\circ\alpha(B)$ (or equivalently that $\alpha(A^2) = \alpha(A)\circ\alpha(A)$). It is a very strong condition to require that there exists a spacetime group G which act automorphically on $\mathfrak{U}$; it is equivalent to the condition that the algebraic structure of a system Ω is invariant under all Lorentz transformations and translations in space and time. From the **active** point of view, we imagine these transformations as defining new descriptions, from a given one, which must be physically realizable. We may imagine the symmetry transformation as acting directly on the physical system, moving the system about in space-time and with respect to space-time orientation (via the *pure* Lorentz transformations or *boosts)*, but with its algebraic structure unchanged.

Obviously the physical interpretation of this statement depends upon the physical interpretation of this algebraic structure; if the latter is assumed to describe the inter-relationships among physical properties (irrespective of value or measure assignments) then we can learn something about what those properties must be (or *cannot* be) in phenomenological terms. If one takes the symmetry group seriously, as a global spacetime symmetry, it is very hard to know whether there is *any* relationship (however indirect) between phenomena, which is in some sense invariant.

In a local sense it is already clear that there can exist no invariant relationship between particle number or "type" and energy, for example. Since particle "type" is associated with the irreducible representations of the space-time group we can neither define our fundamental model in terms of some collection of particles nor in terms of such irreducible representations. On the other hand if we think of free (or weakly interacting) systems (or equivalentlty those states which lead to such a description of the system) then in a phenomenological sense it is clear that these observables

may be useful in the definition of the system, that is, in defining a class of properties invariant under the space-time group in representations determined by such states. If in particular we have a scattering situation it is reasonable to suppose that for perhaps quite a large class of states the system will *always* exhibit the structure of a free system if only we wait long enough. Therefore assuming an assymptotic free particle interpretation gives a way of decomposing the representation of the group in an asymptotic limit. If now the group action is unitarily implementable, one will have obtained the Hilbert space generated (via the GNS construction) by this class of states. Or if one supposes the same reasoning will apply in the remote past of a scattering system, one might assume the (unitary equivalence of) the Hilbert spaces defined by the particle interpretation in the remote past and remote future. If one also assumes unitary equivalence with the Hilbert space which hosts the global symmetry group (and not just some asymptotic part) one might hope to find that the group action is unitarily implementable. This is the assumption of *asymptotic completeness*; it plays a central rôle in the Haag-Ruelle scattering theory. (From the point of view of the general theory it is a highly restrictive assumption; the further attempt to define a unitary evolution in the same representation leads to severe difficulties, cf. Haag's theorem (3.5.3).)

The foregoing considerations arise from an "active" interpretation; it is not obvious that these same implications emerge from a "passive" view of the symmetry transformations (in which one describes one and the same physical system with respect to two coordinate systems connected by a space-time transformation).

There can in fact be no question of a general equivalence between active and passive interpretations of symmetry. This is a troubled and thorny area ; for the sake of definiteness let me take a clear position: an active interpretation of spacetime symmetry supposes that of all possible configurations of a physical system, those which describe

the system in the same way (except with a different space-time orientation and/or in a different region of space-time) can be related by a symmetry transformation. A passive interpretation says that if we have a theory which describes a physical system, and in any such description it is possible to choose one of a class of coordinate systems, then it should be possible to transform from one coordinate system to another, always staying within this class, via a symmetry transformation.

We may think of this as a kind of "re-parameterizing" of each description. But if we ask *why* is it possible to choose one of a class of coordinate systems, there can be many different answers; for example, because a choice of coordinate system is actually a pure artifice, without physical significance, to be defined by convention (physical units, choice of origin, and so on). Or because the physically significant object is a smooth manifold, and a choice of co-ordinates is merely a way of bringing about a local correspondence with $\mathbb{R}^n$, which is a mathematically convenient thing to do (which leads to the general covariance group of general relativity). It may, in certain situations, be possible to find a formalism in which all passively defined symmetries have an active interpretation; or one might characterise those symmetries which have an active interpretation by some other way (this is the difference between writing *e.g.* the generally covariant classical mechanics in the form of special relativity, and defining the space-time symmetries as the isometries of the Minkowski metric). But in general the relationship between active and passive symmetries is not so direct.

There is one further complication; we might also consider the physical system in a fixed configuration, but describe it from different *frames of reference*. We shall refer to this as the *dynamic* interpretation of symmetry. In detail, we consider the various possible configurations of the laboratory system and the question of how one and the same physical system, in a given configuration, appears to each of these laboratory systems. This point of view can be

related to each of the passive or active views, in rather different ways . In particular the set of descriptions so obtained will be in correspondence to the various world-lines accessible to a laboratory system; a certain subset of these will correspond to the inertial frames, and it is reasonably straightforward to argue the equivalence with the symmetries thus obtained with the active symmetries of the spacetime group. That obviously will not apply to gauge symmetries. Dynamic symmetry transformations which interchange *inertial* world lines will be called *kinematic.*

Because of these complications we shall always suppose an *active* interpretation of symmetry transformations[1]. In the following we tacitly suppose that by "space-time group" we mean the inhomogeneous Lorentz group, but in the immediate sequel that need not be assumed; in particular the axioms 1-5 of (2.5.3) do not depend on the symmetry group (so long as it is locally Lorentzian).

To continue with the mathematical developement, it is clear that the existence of an automorphism α on $\mathcal{U}$ will ensure the existence of an automorphism on the associated fundamental model $\mathcal{A},\mathfrak{S}$ by a straightforward transport of structure. Since we assume $\mathcal{U}$ is special we decompose any element A in $\mathcal{A}$ into real and imaginary parts A_+, A_- such that $A_+=\frac{1}{2}(A+A^*)$, $A_-=\frac{1}{21}(A-A^*)$ and define $\alpha(A)$ as $\alpha(A_+) + i\alpha(A_-)$.

This definition is actually more general than that of an automorphism of $\mathcal{A}$, which is defined as for $\mathcal{U}$ but such that $\alpha(AB) = \alpha(A)\alpha(B)$ and with the additional condition that $\alpha(A^*)=(\alpha(A)^*)$ (so that we shall speak of * automorphisms). It is easy to see that if a system is unitarily *or antiunitarily* implemented on $\mathcal{U}$ then it gives rise to an automorphism of $\mathcal{U}$, even though in the latter case it is an *anti-automorphism* on the algebra $\mathcal{A}$ in which $\mathcal{U}$ is embedded. That is, if α on $\mathcal{U}$ is determined by $\alpha(A) = UAU^{-1}$, then an

[1] In (3.2.7) we shall consider Newton-Wigner localization from both active and passive points of view; in (3.5.4) the idea of a "dynamic" interpretation of symmetry will be applied to the Fulling pathology.

arbitrary element of $\mathcal{A}$ will be mapped onto the element $\alpha(A_+)+i\alpha(A_-) = UA_+U^{-1}+iUA_-U^{-1} = U(A_+\pm iA_-)U^{-1}$, depending on whether U is unitary or antiunitary. In the latter case α as a map on $\mathcal{A}$ satisfies $\alpha(A) = UA^*U^{-1}$ and $\alpha(AB) = UB^*A^*U^{-1}=UB^*U^{-1}UA^*U^{-1} = \alpha(B)\alpha(A)$, i.e the induced map on $\mathcal{A}$ is antiautomorphic. An automorphism of $\mathcal{U}$ is called a **Jordan automorphism.**

The action of an automorphism on $\mathcal{A}$ extends to an action on the set of states $\mathcal{S}$ in the obvious way: we define $\hat{\alpha}(f)$ as that state which defines the same values on each $A \in \mathcal{A}$ as does f on $\alpha(A)$. $\hat{\alpha}$ is in fact a bijection on $\mathcal{S}$, and since it is automatically linear on $\mathcal{S}$ it preserves the distinction between pure and impure states.

Any Jordan automorphism is obviously continuous with respect to the algebraic operations; when $\mathcal{U}$ is special it is also continuous with respect to the norm topology (for this and other proofs we refer to Emch [1972]). Because of the duality relation between $\mathcal{A}$ and $\mathcal{S}$ the map $\hat{\alpha}$ is then weakly continuous, and one can actually show that every such map is uniquely associated with a Jordan automorphism. One has a generalisation of the Heisenberg and Schrödinger pictures; they are dual to each other if the time evolution is automorphic on the Segal algebra of the system.

Although the additional generality of the notion of an automorphism at the level of the Segal algebra is welcome, we know that anti-automorphisms at the level of the C^*-algebra $\mathcal{A}$ are only relevent to discrete symmetries so we shall not be conerned with them very much; from now on we assume we have a C^*-automorphism on $\mathcal{A}$ (and by automorphism we will mean a C^*-automorphism).

Given an automorphism on $\mathcal{A}$ we expect to learn something about the action of α on the Hilbert space of the representations of $\mathcal{A}$ determined by the GNS construction. One such theorem is as follows: if a state $f \in \mathcal{S}$ is **invariant** under α then α can be unitarily implemented in the representation π_f , and the cyclic vector of the

representation can be chosen invariant under the corresponding unitary operator. For the space-time groups of automorphisms there is a natural interpretation of such a state; it is the **physical vacuum.**

Consider now the situation when we have a group of automorphisms α_g with the natural continuity requirement that $\langle f;\alpha_g(A)\rangle$ is a continuous function of g for all $f\in\mathfrak{S}$ and $A\in\mathcal{A}$ (recall the analogous assumption in the theory of unitary representations that for U_g acting on $\mathcal{H}$ then $(\phi,U_g\psi)$ is continuous on G for all $\phi,\psi\in\mathcal{H}$). The group continuity property can be extended to a continuity condition on its unitary representation provided when we have the favorable situation of a G-invariant state on $\mathcal{A}$. One then has that each U_g induced in the natural way from each α_g, and the invariant state f used to generate the representation, is in fact *strongly continuous* on $\mathcal{H}_f$. The importance of strongly continuous representations to the general theory of group representations is clear: the Stone theorem relating a unitary operator U on a complex Hilbert space $\mathcal{H}$ to a self-adjoint operator (and therefore to a projection-valued measure) needs at least weak continuity of U on $\mathcal{H}$ (for unitary operators weak and strong continuity coincide). We now see a very simple connection between this special class of group representations and the representations induced by the invariant states on a fundamental model on which G acts as a group of automorphisms.

The set of states in $\mathfrak{S}$ which are invariant under the action of a group G we shall call **the G-invariant states** (denote $\mathfrak{S}_G$); they clearly have a special rôle in the theory of unitarily implementable groups of automorphisms on a system $\mathcal{A}$. An important property of the set of G-invariant states is that it is *compact* in the weak * topology. It is also a *convex space* (the convex sum of any two G-invariant states is obviously G-invariant). The Krein-Millman theorem can then be applied to deduce that $\mathfrak{S}_G$ must have extremal states and that their linear combinations are dense in $\mathfrak{S}_G$ (with respect to the weak *

topology)[2].

The structure of the set $\mathfrak{S}_G$ is of fundamenatal importance for the applications of the theory to statistical mechanics and the study of the thermodynamic limit, as well as a deeper exploration of ideas such as the clustering property of the vacuum. Some of these applications will be discussed in connection to measurement theory (Section 3.5) but we shall not pursue the general theory any further here.

2.5.3. Quantum field theory; general interpretation.

Consider again the general question of how to associate a C^*-algebra with a group G. We have already reviewed one strategy. There is another way of relating a spacetime group to an abstract algebra which brings to the fore the idea of the group as a set of transformations on spacetime, and which assumes that the algebraic structure is tied to spacetime regions in a way which reflects the idea of *causal influence*. The idea arose from quantum field theory, where one has for each subset of spacetime B a set of operators $\Phi(f)$, where f has support in B (the smeared fields); in the present context it is natural to abstain from any specific assumptions about these quantites[3], but to assume that analogous entities exist, the algebraic and norm closure of which is a C^*-algebra associated with the spacetime set B. From an abstract point of view one demands that *a subset of spacetime can be associated with a fundamental model* (we shall call such models **local systems**, to be denoted $\mathcal{A}(B)$). It is then clear that we have a natural action of the group on the set of all local systems, namely that $\alpha_g(\mathcal{A}(B)) = \mathcal{A}(g.B)$, where g.B is the spacetime region onto which the points of B are mapped by the spacetime transformation g, just as we have for the smeared classical or quantum fields (with an obvious limit to the point fields).

[2] Roughly speaking, this is how Segal proved that there must exist a full set of states for a Segal algebra, c.f. (2.3.7).

[3] We are restricted to bounded observables, because only then can we hope to realize an algebraic system as operators on a Hilbert space.

We shall shortly combine the local systems to obtain an algebra, which is also a fundamental model. By construction the group will act automorphically on this algebra; therefore whenever we have a concrete representation generated by a G-invariant state the group will act as a continuous unitary group of transformations on the representaiton space $\mathcal{H}$. We now suppose the spacetime group is a semi-direct product $K\bar{\times}T$. By the imprimitivity theory it follows that there will exist a projection-valued measure P on $\hat{T}$ which will define the representation completely. In this situation if the representation is to have a straightforward physical interpretation *we demand that* P_0 (the restriction of P to timelike vectos in $\hat{T}$) *is a positive operator.* In this way we ensure that the total energy is positive; essentially this is a constraint on the admissable G-invariant states, which have a straightforward physical interpretation (namely: the vacuum).

To further exploit the *local* group action, consider it as imposing a direct relationship between the structure of the field of subsets of spacetime and the structure of the set of subsystems of the total system. At this point the question arises as to whether, and in what sense, such subsystems may be simultaneously and independently described. I take this as central to the problem, of how to define *Cauchy data* for a collection of local systems. Consider then the idea that two local systems can each be described without reference to each other. It is natural to require that this is only possible when the two domains are disjoint; it is also plausible that the domains must be spacelike separated (no time or light-like line intersects both). So we consider the idea that for two such domains B_1, B_2, every observable or property in B_1 can be adequately specified, for the description of all phenomenology in B_1, without reference to the local system with domain B_2 (and *vice versa). Note that correlations between phenomena in* B_1 *and phenomena in* B_2 *are not considered to be assoicated with either domain* (but rather to their union $B_1 \cup B_2$). We now suppose that a *necessary* condition for this notion of

independence is that local systems which are spacelike separated are *compatible*. From the general theory, this means they *commute* (in the sense that every observable associated with B_1 commutes with every observable associated with B_2).

This is a far reaching postulate; it is called **microcausality** [4].

This condition appears necessary but not sufficient to define the notion of the independence of local systems. Clearly compatible observables may not be independent. But it also seems clear that the kind of dependence which is *not* excluded is of the classical kind, that is, when two observables are functions of each other. There is only one *general* way of specifying what kind of functional dependence should obtain between local systems: that is familiar from classical field theory, which is founded on the concept of *local causation*[5]. In a relativistic theory, one simply requires that the fields should satisfy *hyperbolic field equations*. We cannot impose such an expression of causation in quite this way, but we may do something very similar; in some sense we shall requre that a local system associated with a domain B can be completely characterised by data on a **Cauchy surface** for B (that is, a surface which intersects every past directed time-like line which intersects B).

[4] For fermion fields this is guaranteed if the smeared fields anticommute, for bilinear forms in the fields (which generate the observables) then commute. This (stronger) assumption is also called microcausality. So too is the stronger assumption in both fermionic and bosonic theories, that the point fields anticommute and commute respectively. The point fields are not of course observables.

[5] There is an extensive secondary literature which traces the field conept to 18th century concepts of force and the Naturphilosophie movement (e.g. Williams[1965]. On the contrary, I believe the field concept should be traced to Euler and the mechanistic tradition. It is not the fundamental ontology which characterized the emergence of the field concept (of course nowadays the ontology of any field theory is usually considered to be the field itself), but the general strategy of inducing the global dynamics from a local action of the underlying ontology, whatever that is.

This is perhaps too strong; it may be that one needs data on a space-time region which has non-vanishing time-like extent. We shall consider the weaker requirement that the Cauchy problem is well-defined on a **Cauchy volume** for B (that is, as before but with surface replaced by a volume of non-vanishing timelike extent). We must now consider how to express the idea that knowledge of a local system $\mathcal{A}(B_1)$ can provide a complete description of a local system $\mathcal{A}(B_2)$, given that B_1 is a Cauchy volume for B_2. A condition which is certainly sufficient is that $\mathcal{A}(B_2)$ is actually *contained in* $\mathcal{A}(B_1)$, for then every pure state (*i.e.* maximal description) or $\mathcal{A}(B_2)$ can then be obtained by restriction of a *unique* pure state on $\mathcal{A}(B_1)$. This follows from the theorem (Segal [1947a]) that a pure state of a C^*-algebra $\mathcal{A}_1$ is the restriction of a pure state on $\mathcal{A}_2$ whenever $\mathcal{A}_1$ is a sub-algebra of $\mathcal{A}_2$; in this case, it is unique. If we consider that $\mathcal{A}_1$ causally determines $\mathcal{A}_2$ if and only if there is an injection from $\mathcal{A}_2$ to $\mathcal{A}_1$, we obtain from this theorem a subtle phrasing of the intuitive notion of causal determination; that if Ω_1 causally determines Ω_2 then any complete description of Ω_2 can be obtained by a unique complete description of Ω_1. It does *not* say that any complete description of Ω_1 will provide a unique complete description of Ω_2. Of course we know that it must not do so, consistent with what is known about the relationship of systems and subsystems in quantum mechanics. In general a pure state of a compound system will not restrict down to a pure state on a subsystem at all[6].

Adopting this point of view, we are led to postulate that if B_1 is a Cauchy volume for B_2, then $\mathcal{A}(B_2) \subseteq \mathcal{A}(B_1)$. We shall call this the principle of **primitive causality** following Haag and Schroer [1962]. As these authors observe, this principle is *not* satisfied by the generalized free fields of Greenberg [1961] (which satisfy the Gårding-Wightman axioms)

[6] It is curious that this feature of quantum mechanics is sometimes referred to as holism; whilst the knowledge of the parts yields the whole, knowledge of the whole does not yield knowledge of the parts (knowledge in the sense of maximal information).

and is therefore independent of the postulate of microscausality[7].

We are almost ready to formulate a general axiom scheme; it only remains to demand that the algebra of the field of subsets of space-time be incorporated in the obvious way: that the local system associated with $B_1 \cup B_2$ is generated by the set-theoretic union of the local systems of B_1 and B_2 (the norm closure of the algebraic closure of the set theoretic union $\mathcal{A}(B_1)$ and $\mathcal{A}(B_2)$). Since, however, the principle of causal determination is not standard in the literature, we consider the following weaker version of it, which is also a natural transport of structure from the algebra of the B's to the $\mathcal{A}(B)$'s; namely, that if $B_1 \subseteq B_2$, then $\mathcal{A}(B_1) \subseteq \mathcal{A}(B_2)$. This property is called **isotony**; it is an axiom of the Haag-Kastler scheme[8].

In summary we have:

[7] I overstate the case; actually Haag and Schroer obtained these results for a weaker version of this prinicple and when the local systems are assumed to be von Neumann. The weakened version is as follows: for any "time-slice" T_τ of infinite spatial extent and arbitrarily small time-like extent τ then $\mathcal{A}(T_\tau) = \mathcal{A}(T_\infty)$, i.e. complete information on the fields at all times follows from information on the fields on a time-slice. In fact they also showed that with minor techical assumptions primitive causality follows from this condition together with microcausality. Therefore, at the very least, when the local systems are von Neumann we expect that primitive causality is also violated by the generalized field theories; there is a complication with formulating the weakened condition in the present context: the algebra of the form $\mathcal{A}(T_\tau)$ is not a local system, since T_τ has infinite extent.

[8] Isotony is a very natural assumption; it follows from the principle of causal determination because if $B \subseteq B'$ then B' is a Cauchy volume for B.

Axioms for a quasi-local algebra

1. For each open bounded subset B of $\mathbb{R}^4$ there exists a fundamental model $\mathcal{A}(B)$, called a *local system.*

2. The set theoretic union and inclusion relationships of $B_i \subseteq \mathbb{R}^4$ transfer to the local systems: if $B_1 \subseteq B_2$ then $\mathcal{A}(B_1)$ is a sub-algebra of $\mathcal{A}(B_2)$, and if $B_1 \cup B_2 = B$ then $\mathcal{A}(B)$ is the norm closure of the algebra generated by $A(B_1)$ and $\mathcal{A}(B_2)$. We form the C^*-inductive limit $\mathcal{A}$ as the norm closure of the union of *all* local systems. This is called the *quasilocal algebra.*

3. The local systems transform covariantly, that is, if a spacetime group G acts automorphically on the quasilocal algebra, then $\alpha_g(\mathcal{A}(B)) = \mathcal{A}(g.B)$.

4. Microcausality: if B_1 is spacelike separated from B_2, then the observables in $\mathcal{A}(B_1)$ commute with the observables in $\mathcal{A}(B_2)$.

These axioms are called the *Haag-Kastler* axioms; we shall consider the following additional postulate as a natural suplement[9]:

5. Causal determinateness: for locally Lorentzian G if B_1 is a Cauchy volume for B_2, then $\mathcal{A}(B_2)$ is a sub-algebra of $\mathcal{A}(B_1)$.

And finally with regard to the representation theory we demand:

6. The physical vacuum: for G a semi-direct product and $f \in \mathcal{O}_G$ the physical vacuum then P_0 is a positive operator on $\mathcal{H}_f$, where P is the projection valued measure on $\mathcal{H}_f$ associated with the continuous representation of G on $\mathcal{H}_f$ and $\mathcal{H}_f$ is the Hilbert space of the GNS representation of $\mathcal{A}$ generated by f.

It should be noted that, although it is not immediately obvious from the foregoing, the quasi-local algebra does not include (bounded functions of) global observables of the

[9] We distinguish the latter two postulates both because they are not included in the Haag-Kastler axioms and because they depend critically on the spacetime group G.

system (what in conventional field theory would be global integrals of local polynomials in the fields, the total charge, mass, energy *etc.*). These can be affiliated with the quasi-local algebra, in any representation, by forming its bi-commutant (*i.e.* they lie in the associated von Neumann algebra), or simply introduced through the group action on $\mathcal{A}$ as above for those representations which lead to a continuous unitary representation of G. From a realist point of view there is no reason to consider such observables meaningless; it is not an *advantage* of the Haag-Kastler that (bounded functions of) such observables are not themselves in the quasi-local algebra.

The axioms (1)-(4),(6) are satisfied for all the free fields, by the generalized free fields in the sense of Greenberg [1961], by $\lambda\varphi^4$ in 2+1 spacetime dimensions, and by any model which satisfies the Wightman-Gårding axiom scheme (see *e.g.* Glimm and Jaffe [1981] for references[10]). (5) is easily proved for free fermionic fields, and it seems quite certain that it is satisfied by the free field theories and by $\lambda\varphi^4$. One is, however, some way short of being able to establish the existence of covariant quantum fields from the above axioms. In this respect one is a long way removed from the specificity of the concrete postulates of other axiom shemes of quantum field theory (the Osterwalder-Schrader (Euclidean) axiomatization or the Gårding-Wightman axioms). The concept of the quantum smeared field as a covariant operator-valued map on (bounded, open) subsets of $\mathbb{R}^4$ has been replaced by the idea of an abstract algebra of observables covariantly associated with such subsets of $\mathbb{R}^4$. This makes some aspects of the usual formalism -the use of smeared fields and the Wightman reconstruction theorem in particular - appear much more natural. Of course, the algebraic theory also explains why we have to deal with operators on a Hilbert space in the first place.

[10] The formulation of (6) is unorthodox, but I believe consistent with the usual axiom-schemes which assume a Hilbert space is given ab initio.

2.5.4. Historical background to the Haag-Kastler axioms.

The Haag-Kastler axiom scheme was proposed consequent to the realisation that algebraic methods might assist or eliminate altogether the difficulties associated with the so-called "strange" or "myriotic" representations of the Weyl relationships and the algebra of the ACR's for quantum fields, associated with the failure of the Stone-von Neumann-Mackey uniqueness theorem for systems of infinitely many-degrees of freedom. These representations were first discussed in a physical context by van Hove in 1953. He showed that the Hilbert space of the "bare" vacuum of a meson model (with external c-number fields) does not exist in the Hilbert space of the interacting field defined by the "dressed" vacuum (that is, the two vacuum states define inequivalent representations via the GNS construction). We shall discuss the van Hove model, and Haag's theorem (to which it is intimately related) in (3.5.3), to which we refer for further historical background. Suffice it to say here that the initial hope- amongst those working in the foundations of quantum theory and physicists proper - was that such representations were devoid of physical content.

For the more ideologically inclined, the hope was in particular that Fell's result ((2.3.9)) establishing the *weak equivalence* (or so-called *physical equivalence*) of *every* faithful representation of a C^*-algebra could be exploited within an algebraic form of quantum field theory to subsume all these unitarily inequivalent representations. As we have seen, this idea receives support from the positivist interpretation of the weak * topology (in terms of the limitation of any given experiment to a finite number of observables). This same philosophy was also used to motivate the restriction to *local* observables, on the grounds that no actual experiment "refers to" infinite domains in space-time.

This idea was connected to the positivist interpretation of the local system which was (and is) pervasive in the

foundational study of quantum field theory: that a local system $\mathcal{A}(B)$ *defines and is defined by the set of all possible laboratory operations performed in the region* B. This interpretation is independent of the algebraic approach; it is equally applicable to the set of all Wightman functions, restricted to a space-time domain, or in a more figurative sense to (any polynomial of) the smeared fields themselves. For a radical developement of the theory in these terms see Araki [1969].

This interpretation is a straightforward transposition of the dominant operationalist interpretation of quantum theory to the QFT; exactly the same criticisms apply (2.2.1) and I shall not repeat them again.

On the basis of Fell's result, Haag and Kastler postulated that the quasi-local algebra $\mathcal{A}$ has a *faithful irreducible* representation. (Note that there always exist faithful representations; the universal representation (2.3.10) is always faithful. Of course, it is highly reducible.)

2.5.5. Nets and local fields.

The map $B \longrightarrow \mathcal{A}(B)$ from the open bounded sets $\mathcal{B}$ of $\mathbb{R}^4$ to $\mathcal{A}$ is called a **local net**. A general philosophy, independent of the positivist interpretation of the local systems, was widely canvassed by Haag and Borchers in particular, to the effect that a specific choice of local net is analogous to a choice of *co-ordinates* on a manifold. In particular a choice of (covariant, causal) quantum fields on $\mathbb{R}^4$ determines a local net, and is itself analogous to a coordinate system.

This idea arose in the context of the Borchers classification of certain (equivalence classes) of local systems; the theory is based on a certain equivalence relationship, defined by the *locally* unitarily implementable isomorphism of two representations of the quasilocal algebra. He hoped to show that corresponding to each such class (Borchers class) there exists exactly *one* vacuum representation (that is, the Hilbert space of which

contains a state invariant under all translations) (see e.g. Borchers [1967]). There are in general several different ways of defining any one class by a specification of quantum fields (for example from a scalar field one can construct a local net from the Wick-ordered polynomials in the field, its derivatives, and thereby associated (vector or tensor) fields, and any one of them will generate either the same local net or a subnet of it). At a slightly more subtle level the fields are defined as maps, not from the open bounded subsets of $\mathbb{R}^4$ but on a "test" function space which is defined in such a way as to facilitate the very difficult analytic problems so characteristic of the subject. The intricacy and apparent arbitrariness of this construction should, it seems, be understood as reflecting the difficulty and perhaps conventionality involved with "coordinatizing" the quasi-local algebra (see also (3.5.2)).

Working the other way, one can ask whether there is any general construction from a given representation of a net $B \longrightarrow \mathcal{A}(B)$ of all possible local fields which generate the net acting on the Hilbert space of the representation. Knowing this, one will have the "parameters" with which the structure of the representation can be explored; one might at the same time obtain a classification of the representations or information on the structure of the net.

This programme is speculative; to make contact with physics, and in particular with the local couplings of the Lagrangian theory, it seems we actually need to recover the point fields, with all the attendent mathematical pathology.

What is a simple intuitive picture? If the local algebras could actually contain the smeared fields one might hope to show that as B contracts to a point $x \in \mathbb{R}^4$, $\mathcal{A}(B)$ converges in some sense to an algebra which may be generated by a finite set of point fields at the point x. The known properties of the point fields make such a picture rather implausible; for example what is the meaning of the increasingly uncontrolled behaviour of the smeared field as its region of definition is contracted to a point? Why is the high energy

behaviour so singular? One cannot in general define any bounded operator from the quadratic forms and even more singular objects associated with polynomial local couplings in the point fields. *If* however one restricts attention to quadratic forms on the Hilbert space of the representaion, which are of at most polynomial growth in their high-energy behaviour, one can prove an association with the observables of $\mathcal{A}(B)$ where B is any neighbourhood of any point x. This result was recently obtained by Fredenhagen and Hertel [1981]; they were also able to show that for a reasonable class of operator-valued distributions one can always define such a quadratic form together with a natural action on the test function space of the distribution so that the two coincide (the "reasonable class" includes the free massive fields and the super-renormalizable interacting fields in low dimensions). The association with the elements of $\mathcal{A}(B)$ then poceeds by way of affiliation to the associated von Neumann algebra $\mathcal{A}(B)''$.

It is an open question as to whether such methods have more general applications.

2.5.6. Quantization of linear classical systems[11].

We now consider a special class of quantum systems which are defined by a **linear classical system**, that is, a real-linear symplectic space M equipped with a non-degenerate antisymmetric biliear form ω (the symplectic form). In this way, in the finite dimensional case we obtain the familiar free particle theory, and in the infinite-dimensional case the free field theory. The construction is, however, rather involved, so it might be helpful to give a quick summary of the various steps, and the relationship of the theory to the general theory of quantization, in advance of the detailed development.

[11] The remainder of this section is a straightfoward review of the Segal theory of quantization, to be found in Segal [1959a], [1961], [1962]. See also Segal [1967a] for a heuristic exposition of the theory.

The strategy is as follows (I consider only the Bose case in the immediate sequel); we first define a C^*-algebra, called the **abstract Weyl algebra over** M,ω, and which we consider the abstract quantum system corresponding to the given classical one. We next consider a special class of representations for this algebra, which are defined whenever there exists a certain geometric structure connecting M and ω, namely a **complex structure** J defined as a linear map $M \longrightarrow M$ such that $J^2=-1$, which is **compatible** (so that $\omega(Ju,Jv) = \omega(u,v)$ for all $u,v\in M$, in other words so that J is a *canonical* transformation), and which is **positive** (which means that $\omega(Ju,u)$ is positive for all $u\in M$). In the infinite-dimensional case the space M, equipped with this complex structure, is pre-Hilbert (with sesquilinear bilinear form $\omega(Ju,v) - i\omega(u,v)$), and completion with respect to this norm defines the one-particle subspace $\mathcal{K}$ of the Hilbert space $\mathcal{H} = \mathfrak{F}(\mathcal{K})$ of the representation. The phase space of a linear classical field is thus directly associated with the one-particle Hilbert space of the corresponding quantum field. In particular the complex structure, with respect to which $\mathcal{K}$ (and hence $\mathcal{H}$), is a complex vector space, is defined by J. We shall actually obtain the relationship between $\mathcal{K}$ and $\mathcal{H}$ by defining *creation and annihilation operators* from the Weyl algebra and this complex structure, and using the canonical construction of (1.3.4). This representation must also be associated with the GNS representation defined by some state $f\in\mathcal{G}$ on the abstract algera; f is the *vacuum* state. This state can be defined abstractly (and all such vacuum states are in one-one correspondence with the set of all positive compatible complex structures on M,ω). One could say that the positive compatible complex structure on M is used to define a certain state, the vacuum, and that in this way we build a particle interpretation into the theory. Whether some other kind of particle interpretation is possible is obviously an open question; the standard particle interpretation of the free scalar and spin-half field theories is *this* one. These concrete theories are actually shown to define a *canonical* second quantization of a one-particle theory (see Section 3.4).

We gain a greater perspective on this theory if we consider also the general theory of quantization (the so-called geometric quantization). This theory provides a general class of solutions to the Dirac problem: for a classical system M,ω, for any observables f,g (defined as C^∞ functions on M), find corresponding operators $\hat{f},\hat{g}$ acting on a complex Hilbert space $\mathcal{H}$ such that $[\hat{f},\hat{g}] = -i\hbar[f,g]\hat{}_{PB}$ where $[.,.]_{PB}$ is the Poisson bracket defined from the symplectic form ω. This condition is actually a local form of the Weyl relationship for the abstract Weyl algebra in the special case in which f and g generate *constant* Hamiltonian vector fields on M. We shall examine this relationship shortly. The existence of a (positive, compatible) complex structure on M appears as a special case of a more general requirement, that there exists a *Lagrangian subspace* in M. There are in fact two important cases; the one discussed, when this subspace is *Kahler*[12], and the pre- Hilbert space is the space of all *holomorphic* functions on M with respect to the complex structure (Kähler subspaces and compatible complex structures are in one-one correspondence); and the real polarizations of M, where the pre-Hilbert space is essentially the space of all unconstrained functions restricted to the orthogonal complements of these real polarizations (typically the unconstrained functions of position, *or* the unconstrained functions of momentum[13]). In

[12] We shall not need precise definitions of these notions, since we do not use them. Intuitively a Lagrangian subspace of a symplectic space M is a subspace of half the dimensionality such that the symplectic form acting on any two vectors in this subspace vanishes (e.g. configuration space or momentum space as subspaces of phase space). A Kahler subspace is a complex Lagrangian subspace N (we now assume M is endowed with a complex structure) such that $M = N \oplus \bar{N}$ and $N \cap \bar{N} = \emptyset$. One may then recover the complex structure on M given the Kahler subspace. There is a beautiful theory inter-relating these geometrical constructions with the quantization theory, but the mathematics rapidly becomes too sophisticated for the level of this thesis and we shall not persue it. See e.g. Woodhouse [1980] for a self-contained exposition.

[13] Unconstrained, that is, apart from the mild L^2 integrability condition.

this case the connection between the complex structure of the Hilbert space and the geometry of M,ω is not so direct. The holomorphic functions on M, when M is an infinite-dimensional symplectic space, have a natural correspondence with the vectors in Fock space, provided by a power series expansion of the holomorphic function.

This correspondence will be considered in more detail in (3.4.2).

2.5.7. The Weyl relationships.

We recall the concrete Weyl algebra (Schrödinger representation) for a system of 2n degrees of freedom that we wrote down in (2.4.5); we obtained the commutation relationships:

$$[U(\mathbf{s}_1),U(\mathbf{s}_2)] = [V(\mathbf{t}_1),V(\mathbf{t}_2)] = 0,$$

$$U(\mathbf{s})V(\mathbf{t}) = e^{i(\mathbf{s}.\mathbf{t})/\hbar}\, V(\mathbf{t})U(\mathbf{s}).$$

Here the U's and V's are unitary (weakly continuous) representations of the additive group on $\mathbb{R}^n$, $\mathbf{s},\mathbf{t}$, vectors in $\mathbb{R}^n$. From a group theoretic point of view it is natural to interpret the exponential in terms of the characters (or elements of the dual $\hat{A}$) of the additive group on $\mathbb{R}^n$, that is to make the correspondence $\mathbf{t} \longrightarrow \chi_{\mathbf{t}} \in \hat{A}$ where $\chi_{\mathbf{t}}(\mathbf{s}) = e^{i(\mathbf{s}.\mathbf{t})/\hbar}$ and regard the V's as unitary-valued maps on $\hat{A}$. In that case the last relationship may be written as $U(\mathbf{s})V(\chi_{\mathbf{t}}) = \chi_{\mathbf{t}}(\mathbf{s})V(\chi_{\mathbf{t}})U(\mathbf{s})$, and the unitary operators $W(\mathbf{s},\chi) = U(\mathbf{s})V(\chi)$ then obey

$$W(\mathbf{s}_1,\chi_1)W(\mathbf{s}_2,\chi_2) = [\chi_1(\mathbf{s}_2)]^{-1} W(\mathbf{s}_1+\mathbf{s}_2,\ \chi_1\chi_2) \qquad (1)$$

i.e. furnish a projective representation of the additive group $A\times\hat{A}$. In this connection, the significance of the appearance in Eq.(1) above of the exponential $e^{-i(\mathbf{s}_2.\mathbf{t}_1)/\hbar}$, in which $(\mathbf{s}.\mathbf{t})$ is a *symmetric* bilinear form, reflects the fact that we have a duality relationship between A and $\hat{A}$, that it does not matter which one we choose (of the U's and V's), so long as we make *some* choice.

If we think about the space $A\times\hat{A}$, and regard $\hat{A}$ as the cotangent space to A, it is clear that we might be able to generalize the Weyl relationships if we consider $A\times\hat{A}$ as a

the classical phase space of a linear dynamical system. We might even be able to make sense of the Weyl relationships when this space M has a non-trivial topology, and for one-parameter transformations on the cotangent bundle, and in that way obtain a quantum theory which can include constraints - and also free the theory from the use of rectilinear co-ordinates.

Viewed in this way the Weyl relationship Eq.(1) involves a specific co-ordinate system on the co-tangent bundle - the distinction between position (**s**) and momentum (**t**) co-ordinates in physical terms. But consider instead the unitary operators $W(\mathbf{s},\mathbf{t}) = U(\mathbf{s})V(\mathbf{t})e^{-(i/2\hbar)(\mathbf{s}.\mathbf{t})}$; the W's obey the commutation relationships:

$$W(\mathbf{s},\mathbf{t})W(\mathbf{l},\mathbf{m}) = e^{i[(\mathbf{m}.\mathbf{s})-(\mathbf{t}.\mathbf{l})]/2\hbar}\, W(\mathbf{s}+\mathbf{l},\mathbf{t}+\mathbf{m}) \qquad (2)$$

The term in the exponential is typical of the symplectic form acting on vectors of a syplectic space, when one makes an explicit distinction between momentum and position co-ordinates (when one chooses a symplectic basis on M). From the point of view of quantizing a given classical system, it is natural to suppose one is already given a classical phase space M which is a symplectic space, and not just a cotangent bundle, coming equipped with a symplectic bilinear form ω. In that case one is tempted to see the Weyl relationships as a special case of the more general relationship

$$W(u)W(v) = e^{i\omega(u,v)/2\hbar}\, W(u+v) \qquad (3)$$

with u,v vectors in M; the special case of Eq.(2) will result from a choice of a symplectic co-ordinate basis on M. The antisymmetry of ω obviously ensures that Eq.(3) provides a unitary representation of the additive group on each one-dimensional subspace of M (that is that $W(au)W(bu) = W((a+b)u)$). Otherwise we have a projective representation of the additive group on the whole of M. (It is also clear that the restriction to *linear* classical phase spaces plays a crucial rôle in this interpretation of the Weyl relationships; one can, however, make some progress even in the non-linear case (see *e.g.* Segal [1967b], [1970]); here I shall consider only the linear case).

It should be borne in mind in what follows that we have effected a switch from the symmetric bilinear form (s.t) on $A\times\hat{A}$ to the anti-symmetric bilinear form $\omega(u,v)$ on $M = A\times\hat{A}$; the latter is now to be taken as fundamental.

We would now like to free the defining relationship Eq.(3) from its dependence of any particular choice of Hilbert space, starting instead from an abstract version of Eq.(3), and a given classical system M,ω. It is obvious that the W's form a non-commutative associative algebra $\mathcal{A}$ and that the map * defined as $W(u)^* = W(-u)$ is anti-automorphic on $\mathcal{A}$. So we have a * algebra. But the idea of weak continuity is not a pure C^*-algebra concept. To be sure, if we are given an abstract C^*-algebra and a continuous symmetry group of automorphisms of the algebra (defined as in (2.5.2)) we know this group will act in any representation space $\mathcal{H}$ by weakly continuous operators (that is continuous with respect to the weak operator topology in $\mathbf{B}(\mathcal{H})$); it is exactly because we want to go the other way that we have to characterize the idea of weak continuity in a representation free way.

Before pursuing this question, let us first see what happens if we assume we *do* have a representation of the Weyl relationships, in which the W's are weakly continuous unitary operators on some Hilbert space $\mathcal{H}$. In that case we shall say we have a (**concrete**) **Weyl system** over M,ω. So far we know we have a Weyl system in the finite dimensional case, which is the Schrödinger representation (or a direct sum of these). So now we suppose there is a Weyl system for an infinite dimensional M. In the following we shall make no use of the actual representation space of this concrete system; we might be talking about any concrete representation in which the W's are weakly continuous, that is any Weyl system.

In fact we shall work with what should be self-adjoint operators in any representation of the abstract algebra, namely the generators of the W's (the whole point of weak continuity is of course that we can define self-adjoint operators, with natural domains of definition, from Stone's

theorem in this way). So we suppose that we have a representation of the abstract algebra in which for each $u \in M$ there will exist a self-adjoint operator $A(u)$ which is the generator of the one parameter group $W(su)$ so that $W(su) = e^{-isA(u)/\hbar}$. We begin by considering their commutation relationships, which follow from Eq.(3):

$$[A(u),A(v)] = -i\hbar\,\omega(u,v). \qquad (4)$$

In terms of these self-adjoint operators, we can hope to make aconnection with the Dirac quantization rule. Eq.(4) is similar to this rule, except that the symplectic form and not the Poisson bracket occurs on the RHS. We recall the general relationship between the two objects. Let N be a 2n-dimensional symplectic *manifold* and consider the class $\mathfrak{C}^\infty(N)$ of real -valued infinitely differentiable functions upon it. These are called the **classical observables** of N. Every $f \in \mathfrak{C}^\infty(N)$ is at the same time a smooth map on phase space into the reals, and the generator of a vector field X_f on N, defined via the symplectic form σ on N:

$$\sigma(X_f, Y) + Y(f) = 0$$

for all vector fields Y on N. This vector field is canonical in the sense that it preserves σ, that is the Lie derivative of σ with respect to X_f vanishes, and the (locally) defined integral curves of this vector field define local one-parameter families of canonical transformations of N. If these vector fields are globally defined they are called **globally Hamiltonian** (denote HV(N)); and vector fields which preserves σ in the above sense are **locally Hamiltonian** (denote LHV(N)). LHV(N) is a Lie algebra under the Lie bracket, since it is well known (Sternberg 1964) that if $Y,Z \in$ LHV(N), then $[Y,Z] = X_{\sigma(Y,Z)}$ (that is, the locally Hamiltonian vector fields are closed under the Lie bracket [.,.]).

The **Poisson bracket** of $f,g,\in\mathfrak{C}^\infty(N)$ is the C^∞-function $[f,g]_{PB}$ on N defined by $[f,g]_{PB} = X_f(g) = -X_g(f) = \sigma(X_f,X_g)$. (Therefore for any $f,g \in \mathfrak{C}^\infty(N)$ it follows that $[Y_f,X_g] = X_{[f,g]_{PB}}$, which makes it clear that the algebra of classical observables with the product given by the Poisson bracket, is homomorphic to the Lie algebra defined by LHV(N)).

For us the key point is that when N is a linear vector space then the *constant* vector fields on N can be identified with elements $u \in N$. For *those functions* in $\mathcal{E}^\infty(N)$ which generate constant vector fields on N (that is linear C^∞-functions on N) we can just identify each f with a vector u in N (so we shall write $u_f \in N$ instead of $X_f \in$ LHV(N)). Dirac's quantization rule, which becomes $[\hat{f},\hat{g}] = -i\hbar[f.g]^{\wedge}_{PB} = -i\hbar\omega(u_f,u_g)^{\wedge}$, is then equivalent to the commutation rule (4) if we define $\hat{f} = A(u_f)$, for linear f on N.

This may appear unduly restrictive; we shall only obtain quantum analogues of the *linear* observables on M. But in fact these provide a large enough class of observables (via their algebraic combinations) for our purposes. In Section 3.4 we shall obtain models with infinite dimensional M, where the linear functions on M are defined by a space of test functions $\mathcal{T}$, with the action $f(u) = \int f(x)\Delta(x-x')u(x)d^4x$, $f\in\mathcal{T}$ (Δ is the causal commutator discussed in (1.4.4)). The quantum analogues of the functions f (the $\hat{f}$'s) are the smeared fields. The elements of $\mathcal{T}$, which define linear maps on M in this way, therefore also define constant vector fields on M, which can be identified with elements u of M. Explicitly, $u(x) = \int f(x')\Delta(x'-x)d^4x'$. The quantum fields defined by the A(u)'s are not quite the smeared fields, but self-adjoint operators associated with solutions of the classical field equations. They are still associated with space-time domains, however (the supports of the u's).

Suppose now that in the infinite dimensional case there is a positive compatible complex structure J on M. It is easy to see that $\omega(J.,.)$ is a *symmetric* bilinear form on M. In fact $\omega(Ju,v)=\omega(J^2u,Jv)$ (since J is canonical) $= \omega(-u,Jv) = \omega(Ju,v)$ (by antisymmetry and linearity). Consider now the expression

$$\langle u,v\rangle_J = \omega(Ju,v) - i\omega(u,v). \qquad (5)$$

We have that $\langle u,bJv\rangle_J = b\omega(Ju,Jv) - bi\omega(u,Jv) = ib\langle u,v\rangle_J$ (so that $\langle .,.\rangle_J$ is *complex linear* in its second factor). On the

other hand $\langle bJu,v\rangle_J = b\omega(J^2u,v) - b\omega(Ju,v) = -ib\langle u,v\rangle_J$ so *that it is complex anti-linear* in its first factor); it follows that $\langle .,.\rangle_J$ is a *sesquilinear form* on M, regarded as a complex linear vector space with the action of $\mathbb{C}$ on M given by $(a+bi)u = (a+bJ)u$. Since we suppose J is positive, this form is positive definite, so we actually have a pre-Hilbert space. We shall write M_J for the completion of this space.

Now consider the operators $A(u)$; since $\omega(u,u)$ is zero for any u it follows that $A(.)$ is a *real linear* map on M. Therefore we may define the operators:

$$b(u) = (2\hbar)^{-1/2}(A(u) + iA(Ju))$$
$$b(u)^* = (2\hbar)^{-1/2}(A(u) - iA(Ju)) \qquad (6)$$

It is obvious that $b(u)^*$ is indeed the adjoint of $b(u)$; we can evaluate their commutation relationships from that of the A's. Clearly $[b(u),b(v)] = [b(u)^*,b(v)^*] = 0$ and

$$[b(u),b(v)^*] = \frac{1}{2\hbar}\big([A(u),A(v)] + [A(Ju),A(Jv)] + i[A(Ju),A(v)] - i[A(u),A(Jv)]\big)$$
$$= \omega(u,Jv) - i\omega(u,v) = \langle u,v\rangle_J. \qquad (7)$$

We have met these commutation relationships before ((1.3.4)); they are CCR's for the creation and annihilation operators $b^*(u), b(u)$, which create and destroy (respectively) a particle in the state u, an element of a complex Hilbert space with inner product $\langle .,.\rangle_J$. In (1.3.4) we considered an arbitrary Hilbert space, but expressed the creation and annihilation operators concretely; this action of course implies the commutation relationships above, so we may take over this action directly in the present case as *one* representation of these commutation relationships. That is, we may define the concrete representation of the b's on M_J by constructing the Fock space over M_J, $\mathcal{H} = \mathfrak{F}(\mathcal{H}) = \mathbb{C}\oplus M_J\oplus M_J\otimes M_J\oplus\ldots$ and proceed as in the canonical theory of (1.3.4). By construction (what would be) the one-particle subspace of $\mathcal{H}$ is here the complexified symplectic manifold M_J. Note that consistent with this action (*cf.* Eq.(39)(1.3.4)), $b^*(.)$ is automatically complex linear, and $b(.)$ complex *antilinear*, on M_J (since $b(Ju)^* = (2\hbar)^{-1/2}(A(Ju)$

$-iA(J^2u)) = ib(u)^*$, $b(Ju) = (2\hbar)^{-1/2}(A(Ju)+iA(J^2u))= -ib(u))$. This fact will be of great importance in Section 3.4.

In all respects every occurrence of complex numbers a+ib in the non-relativistic theory is given in the present theory by a+Jb; we expect to find that in the non-relativistic theory multiplication by i is indeed a positive compatible complex structure on M. We shall see this explicitly in Section 3.4; we shall also see that this is *not* the case for the relativistic theories.

In this way we can construct a Fock represesentation for the Weyl system over M,ω; it is apparent that all that we have used to do so is the complex structure J. But we have no idea how general this representation is; neither have we exploited the algebraic representation theory in this construction. Also, we have so far conducted everything on the assumption that we do have a Weyl system, that is a concrete representation, which obviously leads as we have just seen to a new concrete representation on $\mathfrak{F}(M_J)$, on the assumption that J exists. Let us therefore turn to the problem of defining an abstract version of a Weyl system.

2.5.8. The abstract Weyl algebra; regular states.

The abstract Weyl algebra is defined as follows: we consider a concrete Weyl system, which we shall denote $\mathcal{W}$, and from it construct a certain (concrete) algebra $\mathcal{M}$; this has to be a C^*-algebra, but otherwise the only constraint is that it must include all the W's provided by $\mathcal{W}$. We want to show that $\mathcal{M}$, as a C^*-algebra, is actually independent of the particular $\mathcal{W}$, so it is desirable to use a small algebra $\mathcal{M}$, which has a simple definition (obviously independent of the representation space of $\mathcal{W}$). We take the norm completion of this algebra, with respect to the states on $\mathcal{M}$ defined by the density matrices on $\mathcal{W}$, and the resulting C^*-algebra turns out to be independent of $\mathcal{W}$. Likewise the states on $\mathcal{M}$ which can be obtained in this way (*i.e.* as vector states of some representation in which the W's are weakly continuous)

are defined independent of any particular W (these are called the **regular states** of $\mathcal{M}$, denote $\mathfrak{S}_{reg}$). The final abstract definition of a Weyl algebra is then the pair $\mathcal{M}, \mathfrak{S}_{reg}$. We can be assured that, in the GNS representation corresponding to any state $f \in \mathfrak{S}_{reg}$, the unitary operators representing the W's will be weakly continuous. It also follows that $\mathfrak{S}_{reg}$ is a convex set, weakly closed and compact in the dual * topology, so all the general theory goes through for this class of states.

I shall merely state the results of this construction; for details I refer to Emch [1972], Segal [1967a]. It is obviously desirable to prove the existence of a Weyl system, rather than to have to suppose there is such a system, and to this end there is a simple construction which is always possible whenever one can define a positive definite symmetric bilinear form S(.,.) on M, relative to which ω is continuous. That is, $|\omega(u,v)|^2 \leq S(u,u)S(v,v)$. We can then take a class of well-behaved functions on M, and use S to define a probability measure on M (call μ_S); the completion of this space in the L^2 sense forms the representation space, $L^2(M, d\mu_S)$. We then define, for f in this space, the representation of the W's defined by

$$(W(u)f)(v) = f(u+v)e^{i\omega(v,u)/\hbar}e^{(-S(v,u)-S(u,u))/2\hbar}$$

(the last factor is necessary to make the W's unitary). It is easy to check that the W's defined in this way satisfy Eq.(3) of (2.5.7), and are weakly continuous. So we have at least this Weyl system when such an S exists.

Now we suppose there is a Weyl system. To define a reasonable algebra, we use one of the group algebras defined by Segal [1951], but with M infinite dimensional to define functions on M we first consider only a finite-dimensional sub-space $N \subseteq M$. For, say, the C^∞ functions on N of compact support, we define the operators $F = \int_N W(u)f(u)du$ (the measure du is well-defined on N). We then algebraically complete the F's, using the Weyl relationships, and in this way obtain a * algebra $\mathcal{A}_N$ for each N (which is not, however, a C^*-algebra yet). We now take the union of all the $\mathcal{A}_N$'s, making the natural assumption of isotony (*cf.* the

construction of the quasi-local algebra in (2.5.4); this actually means we can no longer use F's defined only by functions on M, but we must use measures on M as well). Denote the resulting algebra by $\mathcal{A}_M$. The density matrices of the Weyl system that we started with define states on $\mathcal{A}_M$, and in the norm derived from these states we complete $\mathcal{A}_M$ to obtain a C^*-algebra $\mathcal{A}_W$. It is then a theorem (Segal [1959a]) that for W_1, W_2 any two Weyl systems on the Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$, over the classical system M, ω, then the C^*-algebras $\mathcal{A}_{W_1}, \mathcal{A}_{W_2}$ are * isomorphic.

This uniqueness theorem is quite general for C^*-algebras which are constructed in this way from algebras over directed sets, each one of which is finite dimensional, and such that there is a unique (equivalence class of) continuous representations of the algebra whenever it is restricted to finite-dimensional sub-sets. The uniqueness of the representations is only needed to make sure we have a purely algebraic characterization of the finite sub-systems. If, for example, we had a directed set and a C^*-algebra for each subset, defined in a representation-free way, and so that the isotony condition is maintained, we always get a unique C^*-algebra (*cf.* Takeda [1955] and note the relevance of this result to the quasi-local construction of (2.5.4)).

In this way we obtain a *unique* abstract Weyl algebra, defined by any Weyl system. Just as important is the fact that the class of regular states, *i.e.* those which generate representations in which the W's are weakly continuous, can also be defined independent of any particular Weyl system. That is, we can define the concept of regularity in such a way that it is preserved under the * isomorphism under which all the concrete algebras $\mathcal{A}_M$ can be identified. This is obviusly true if we just define regular states as above, as those which generate representations in which the W's are weakly continuous. The regular state generating the representation will then of course exist as a cyclic vector state in the Weyl system defined by this representation. So equally we could define the regular states as all those states which can be defined by vector states in (some) representation in which the W's are weakly continuous.

Another abstract definition of regular states is obtained by making use of the uniqueness theorem for the finite-dimensional M's; with this one can define regular states for each $\mathcal{A}_N$ as those which can be given as density matrices in the Schrödinger representation. A regular state of $\mathcal{A}$ is then any state which restricts to a regular state (in this sense) for any finite-dimensional subspace N of M.

A third, more direct definition is actually the most useful. It is expressed in terms of the **generating functional** of a state, that is, the functional $\rho(u)$ defined by any $f\in\mathfrak{S}$ on $\mathcal{A}$, as $\rho(u) = \langle f; W(u)\rangle$. It is intuitively obvious that the concept of regularity expressed in terms of these generating functionals will require that $\rho(u)$ should be continuous on any finite-dimensional subspace of M. Actually to obtain precisely the same notion of regularity as above we need to demand that ρ is also continuous on the F's (that is, $\langle f; \int_N g(u)W(u)du\rangle$ should be continuous on N, for any finite dimensional N, and for any g in $\mathfrak{S}^\infty(N)$ of compact support). All of the requirements on ρ, which ensure it defines a unique state on $\mathcal{A}$ with these continuity properties, are as follows:

(1) $\rho(u) = 1$

(2) $\rho(u)$ is continuous on finite dimensional $N\subseteq M$.

(3) For each finite set $u_1,\ldots,u_n\in M$ and each finite sequence $\lambda_1,\ldots,\lambda_n \in \mathbb{C}$, $\sum_{ij} \rho(u_i-u_j)\, e^{(i\omega(u_i,u_j)/2\hbar)}\bar{\lambda}_i\lambda_j \geq 0$

(this last condition, apart from the "twisting factor" (the exponential term), is that ρ should be a positive definite function as defined in (2.4.4)). We shall use this definition of states in terms of generating functionals later; we note in passing that since ρ fixes a state (of course a regular state) f in $\mathfrak{S}$ it also defines a representation π in which $\rho(W(u)) = \langle f; W(u)\rangle = \langle\phi, \pi(W(u))\phi\rangle$ (where ϕ is the cyclic vector in $\mathcal{H}_\pi$ which is determined by f in $\mathfrak{S}$, and $\langle .,.\rangle$ the inner product in $\mathcal{H}_\pi$). We can make contact with the standard formalism of generating functionals in quantum field theory by observing that $\pi(W(u))$ will be of the form $e^{-iA(u)/\hbar}$ for self-adjoint $A(u)$ on $\mathcal{H}_\pi$ (for this we need that ρ is regular) in which case:

$$(\hbar/i)^n \left\{ \frac{\partial^n}{\partial t_1 \ldots \partial t_n} \sum_{1,j} \rho(t_1 u_1 + \ldots + t_n u_n)\, e^{(it_i t_j \omega(u_i, u_j)/2\hbar)} \right\}_{t_1 = \ldots = t_n}$$

$$= \quad \langle \phi, A(u_1) \ldots A(u_n) \phi \rangle. \qquad (8)$$

In other words, the **n-fold vacuum expectation values** can all be derived from the functional ρ. (we have not stated sufficient conditions for existence).

The fundamental result, that we can define the Weyl algebra and its regular states in an abstract way, leads to an important corollary: any symplectic transformation on M induces a unique * automorphism α of the abstract algebra which carries regular states into regular states. So certainly if the physically relevant transformations on the quantized system can be given as symplectic transformations of the classical phase space we will obtain * automorphisms of the algebra.

Let us consider this result, and the existence question for a Weyl system, from the point of view of the complex structure J. We have seen that existence is ensured if there exists a positive-definite *symmetric* bilinear form S on M; we observe that $\omega(Ju, v)$ is such a form, relative to which, moreover, ω is continuous. So if there is a positive compatible complex structure J on M then there exists a Weyl system and from that an abstract Weyl algebra $\mathcal{A}$ together with a set of regular states $\mathcal{G}_{reg}$. However the converse is *not* in general true; from a positive definite symmetric bilinear form S one cannot construct a compatible positive complex structure J.

What the complex structure does is to ensure that a symplectic transformation on M which preserves J can also be seen as a *unitary* transformation on M_J, and *vice versa.* Combined with the previous result we see that, in this favorable situation, a symplectic transformation on M is at once a * automorphism of $\mathcal{A}$ and a unitary transformation of M_J. As we shall see, the * automorphisms so defined are actually the *canonical second quantization* of the unitary transformations on M_J.

2.5.9. The Fock representations.

From the foregoing we know that for each regular state we can generate, via the GNS construction, a Weyl system. We can also construct the canonical Fock-Cook representation of the algebra of creation and annihilation operators; since the generators of the W's are then explicitly given one also has a concrete representation of the Weyl algebra, and we can determine the corresponding state in $\mathfrak{S}_{reg}$ or generating functional ρ. It turns out that this generating functional is defined directly by the complex structure J and the symplectic form ω; in fact, for the Fock-Cook representation obtained above on $\mathfrak{H}(M_J)$ it is $\rho(u) = e^{-1/4|\langle u,u\rangle_J|^2}$ where $\langle .,. \rangle_J$ is the Hermitian form given by $\langle u,v\rangle_J = \omega(Ju,v) - i\omega(u,v)$. We see that the complex structure enters here in a critical way; the representation is *not* fixed by the symplectic form ω alone. On the other hand we also see that the generating functional has a very special property, namely that it is invariant under *any* unitary transformation on M_J. This is also true of the state in $\mathfrak{S}_{reg}$ with which it is associated. It is physically natural to require that the vacuum state is invariant under the inhomogeneous Lorentz group (as we have seen this will ensure that these transformations are unitarily implementable in the representation determined by this state, *cf.* (2.5.2)); we discover that for the Fock-Cook representation we have a more generally invariant state, invariant under all the "one-particle" unitary transformations on B_J. In this situation, the unitary transformations on M_J are in correspondence with the symplectic transformations on M[14].

[14] If one has a time-evolution defined on M, in particular as a symplectic one-parameter group of transformations on M, we may have several positive compatible complex structures on M, any of which will make M into a complex vector space on which these symplectic transformations are unitary transformations. But the requirement that the generator of this one-parameter group is positive (equivalently, that the total energy operator has a purely positive spectrum) will select a sub-set of this set of complex structures; we shall proceed in this way in Section 3.4 to obtain the standard formalism of the Dirac and scalar fields.

PART THREE: APPLICATIONS TO THE INTERPRETATION OF QUANTUM FIELD THEORY

Introduction.

At the end of Part I certain difficulties of interpretation were proposed that both reflect the preoccupations of a number of the physicists at the time and which, in my view, have continued to contaminate the standard literature on quantum field theory.

These difficulties are resolved in what follows. What is involved is the systematic application of the theory of Part II, in particular Sections 2.3 - 2.5. Sections 3.1 -3.3 will cover ground familiar to the physicist[1]. I do not know of any interpretation of the idea of q-locality or the geometric quantization, other than that to be found in Segal's original papers, which were confined exclusively to the real scalar case. Nor am I aware of an explicit treatment of the complex scalar field. These topics are therefore discussed in some detail (see, in particular, (3.2.8), (3.3.7) and Section 3.4).

The fundamental new feature of the canonical quantum theory, in its applications to systems of infinitely many degrees of freedom, is that there exist inequivalent representations; this is exploited (in a rather implicit way) in Section 3.4 and in Section 3.5 it is explicitly applied to the measurement problem along the lines of Hepp [1970]. In this way our account of the real-abstract philosophy of Part 2 is brought to a conclusion. We also give a discussion of the interpretation of non-Fock representations, in relationship to the field-many-particle equivalence, Haag's theorem, and quantization on curved spacetimes.

[1] Sections 3.1 and 3.2 have a historical bias. In view of the importance of the Wigner paper of 1939, and in the absence of secondary literature on this paper, it is discussed in some detail in Section 3.1.

3.1. The Elementary System

> It seems to me that the deliberate utilization of elementary symmetry properties is bound to correspond more closely to physical intuition than the more computational treatment.
>
> E. Wigner, 1931

3.1.1. Introduction

In the light of Section 2.4 we know how to go about classifying the irreducible unitary representations of the orthochronous part of the ILG; we must determine all irreducible unitary representations L of the stability subgroup G_0 of G at some arbitrary point in the dual to the translations subgroup T of G, and construct the corresponding induced systems of imprimitivity. In this section we shall carry this out explicitly. First, however, in introduction to both this section and the next, it is helpful to review the historical context of the theory.

Wigner first announced his results in an address to the American Mathematical Society in May 1935. In the short note that subsequently appeared in the Bulletin (Wigner [1935]) he emphasised the fundamental idea that "there is a unique correspondence between the possible Lorentz invariant equations of quantum mechanics, on the one hand, and the representations of the inhomogeneous Lorentz group by linear operators, on the other", an idea which he attributed to Dirac, with whom he proposed to write a detailed paper. Both here, and in the paper which eventually appeared some 4 years later, he stressed that his methods imply a gain in

generality as compared with the usual tensor calculus; that one "must obtain in the present calculus all equations, even such (if they exist) in which the coordinate is quantized, for example." (Wigner [1935]). This is a remarkable claim, but in fact Wigner overstated his case: he did not prove this claim, nor did he explicitly obtain any of the known wave equations.

This section and the next provide an account of what Wigner probably meant by this remark; he undoubtedly grasped the notion, which should not be attributed to Dirac, that the relativistic wave equations are "elementary", in a sense which transcends the notion of an irreducible group representation. The perspective that is missing comes from the theory of von Neumann algebras, where we learn that the spacetime groups are so special that the decomposition theory of representations is just like the finite dimensional case, *always* into disjoint irreducible parts, and the latter easily parameterized by the mass hyperbola and a single integer. If I may use a metaphor, it is one thing to establish that an object is an atom (let us say, indivisible), and another to show that the world is atomistic.

For free fields and particles, the relationship between the relativistic wave equations and the Schrödinger equation was the main puzzle left unresolved by the founders of quantum field theory; it is of course just to help solve this puzzle that we have reviewed the Mackey theory in Section 2.4. By 1937, in addition to the configuration space and momentum space forms of the KG and Dirac equations, and the spinor forms of these equations, the general spinor and tensor equations for particles of arbitrary spin had been proposed by Dirac [1936] and Proca [1936], elaborating the pioneering work of Majorana [1932]. Their study was to continue in papers by Duffin [1938], Kemmer [1938], Fierz [1939], Pauli and Fierz [1939], Bhaba [1945], Bhaba and Harish-Chandra [1946], and Harish-Chandra [1946], [1947], to name but the

most important[1]. In 1937 Wigner appeared to consider the relationship between these wave equations and the representations that he constructed trivial; in a somewhat understated paragraph he declated:

> The difference between the present paper and that of Majorana and Dirac lies-apart from the finding of new representations- mainly in its greater mathematical rigor. Majorana and Dirac freely use the notion of infinitiesimal operators and a set of functions to all members of which every infinitesimal operator can be applied. This procedure cannot be mathematically justified at present, and no such assumption will be used in the present paper. Also the conditions of reducibility and irreducibility could be, in general, somewhat more complicated than assumed by Majorana and Dirac. Finally, the previous treatments assume from the outset that the space and time coordinates will be continuous variables of the wave function in the usual way. This will not be done, of course, in the present work. (Wigner [1939 p. 152]).

And that is actually all he had to say about the relativistic wave equations. In fact, whilst he obtained unitary representations of the form

$$(U_g f)(p) = (r_g(g^{-1}.p))^{1/2}\phi(g, g^{-1}.p)f(g^{-1}.p) \qquad (1)$$

(c.f. Th.2.4.1), with ϕ a unitary operator on a separable Hilbert space $\mathcal{K}$ and U a unitary operator on the Hilbert space $\mathcal{H} = L^2(\hat{T}, \mathcal{K}, \mu)$, Dirac had obtained the momentum space wave equations

$$\gamma_k^{\mu} p_{\mu} \psi = m\psi, \quad k = 1,\ldots,N;\ \mu = 1,..,4; \qquad (2)$$

where the ψ's are functions of the momentum and a *symmetric* function of N four-valued variables $\zeta_1,\ldots,\zeta_N$, and the γ's (for each k) obeys the same algebra as the γ matrices of the electron theory. The Hilbert space is the set of all such square integrable functions with respect to the inner product

[1] Unfortunately for lack of space I shall not be able to discuss these developments, which form the essential background to the construction of massive spin 1 covariant wave equtions and spin 1 massive QFT. In view of the emphasis in this thesis on scalar bosonic field theory, this is unfortunate, (from the point of view of the standard model) for to date no elementary scalar boson has been empirically observed, whilst massive elementary spin 1 bosons have been detected (the $W^{\pm}$ and Z^0 bosons of electroweak theory). However the Higg's particle is of great theoretical importance and may yet be observed.

$$(\psi,\phi) = \int |\Sigma_\zeta \bar{\psi}\, \gamma_1^4 \ldots \gamma_N^4 \psi|\, d\mu$$

(here and in the above μ is the invariant measure on the positive mass hyperboloid). From the work of Majorana [1932] it was clear that this wave equation referred to a particle of spin s=N/2; this fact, together with the fact that these representations are unitary, were the basis of Wigner's confidence that the relationship between them and the representations that he had discovered was trivial. They (or perhaps some irreducible subrepresentation) must be unitarily equivalent[2].

But quite apart from determining these subrepresentations, should the Dirac-Majorana equations prove reducible (and they mostly are, because they are representations of the extended ILG, that is including the inversions) there can be profound physical *and mathematical* distinctions between unitarily equivalent mathematical systems. At any rate, Wigner came to write two papers on the subject: the first (Wigner [1947]) a review of the various inner products defined in configuration space and spinor treatments of the wave equations; and the second (Bargmann and Wigner [1948]), in which he showed explicitly that the set of equations Eq.(2) imply that the stability sub-group at the points (mc,0,0,0) defined by this representation is indeed unitarily equivalent to the familiar 2s=N dimensional representation of the rotation group corresponding to spin s. The Casimir invariants of the Lie algebra defined by Eq.(2) were also evaluated and found to be as expected (i.e. the square of the four momentum m and the square of the Pauli-Lubanski four - vector evaluated as $m^2 s(s+1)$); in addition Bargmann and Wigner also established the appropriate connections for the mass zero and continuous spin representations.

However this falls some way short of a complete investigation. One wants an explicit construction for the transformation taking one from the canonical form of Eq.(1)

[2] I shall consider this relationship in the case N = 1 in Section 3.3. This case is, of course, the Dirac equation.

to the covariant wave equations. So far as I know, the first published account is due to Joos [1962], and independently Weinberg [1964], who showed how to pass from the form Eq.(1) to the familiar configuration space fields, in his attempt to free the Feynman rules from any logical dependence on the assumptions of quantum field theory. Transcribed to momentum space, this construction was almost certainly known prior to that time; it is an example of a general class of representations, known variously as vector bundle, Hilbert space bundle, or line bundle representations. A systematic account of this type of representation in the general context of imprimitivity theory may be found in Varadarajan [1970].

There can be no doubt that only a part of this story was known to Wigner in 1939. Nevertheless he correctly perceived that the *elementary system*, defined as an irreducible unitary representation of the spacetime group (or more precisely its equivalence class), must be determined by the (unitary equivalence class) of its restriction to the stability subgroup, which is to say (in the massive case) the value of m^2 and the dimensionality of the representation of SU(2) (the universal covering group of the rotation group SO(3)). As a compact semi-simple Lie group we know that all its irreducible representations are finite dimensional and uniquely characterized (up to unitary or anti-unitary equivalence) by the dimensionality N of the representation, or equivalently by the spin $s = \frac{1}{2}(N-1)$. As Wigner observed, there may yet be a doubling of the dimensionality of this representation space if one extends the representation to an irreducible representation of the discrete symmetries, that is to the entire inhomogenous Lorentz group and not just the orthochronous group. In particular, he pointed out that including the symmetry, which has the action of interchanging the positive and negative mass hyperboloid, always leads to an inequivalent (orthogonal) representation, which corresponds to the states of "negative energy" (but does *not* double the dimensionality of the representation space for the compact stability sub-group).

It was already clear from Wigner's analysis that for positive mass the space-reflection (parity) symmetry did *not* lead to an inequivalent representation (the situation is otherwise for the zero-mass non-zero finite spin elementary system, for example the photon).

The inescapable conclusion is that the Dirac equation, employing 4-component spinors, cannot express only the time-evolution of the state but must also be a constraint equation which reduces the number of independent components to two. This insight seems to have been denied to the many physicists[3] who have interpreted the appearance of 4-components in the Dirac theory in terms of the existence of positive and negative frequency solutions, each with two-independent components (i.e. 2+2=4). I shall discuss this point further in (3.3.6).

3.1.2. The Wigner paper of 1939.

This paper, entitled "On unitary representations of the inhomogenous Lorentz group" and submitted to the Annals of Mathematics in the close of 1937, was the first major attack on the theory of non-compact locally compact groups and contained what turned out to be an exhaustive classification of the irreducible representations of the ILG.

This paper marked the tentative beginnings of the representation theory of such groups and much remained unclear. One reason for this is that he makes frequent reference to a forthcoming paper of von Neumann at crucial points in the argumentation. This paper was never published; however related material appeared some ten years later, consisting of a general theory of the reduction of rings of operators as the seventh of the monolithic series of papers on operator rings; however, it

[3] For example Foldy and Wouthuysen [1950], and Foldy [1956]; Dirac too [1928a] made this mistake, but he did not have the benefit of the Wigner theory.

made no reference to the Wigner paper[4].

The structure of the paper was as follows; the first three sections were concerned with general orientation and interpretation of the material following. It is only here (2 part C) that he makes limited connection with the decomposition theory of operator rings, and then only in summary form. His principal claims were (i) that any ring generated by a given unitary representation of the ILG (that is, the closure of the set of all polynomials in the unitary operators provided by the representation) can be written as the direct sum of factor representations, (ii) that an operator ring which is a factor and generated by a unitary representation of the ILG is of type 1, and (iii) that every type 1 representation of a factor of the ring can be written as the direct sum of irreducible representations of the associated group. Results of this kind had been obtained by Murray and von Neumann [1936] (with the possible exception of (ii)); the general formulation of (ii) for regular semi-direct products is due to Mackey [1957], and is discussed below).

The connection with the wave equations he described as follows; using the by then familiar arguments of Wigner [1931] it was immediately established, if the symmetry group G is to act linearly on a complex Hilbert space $\mathcal{H}$ so as to preserve the inner product, then any (hence every) element of the group will be represented by a unitary operator on $\mathcal{H}$.

[4] Mackey, who edited Von Neumann's complete works, found the original paper in Von Neumann's files after his death. Due to a flaw in the proofs (probably relating to Th. 3.1.1), the paper remained unpublished. It was Mautner [1950] who first systematically applied his direct integral decompositions to the inhomogeneous Lorentz group. We also note that Wigner's methods were heavily reliant on the fact that the ILG is a semi-direct product; much of Mackey [1949b], [1952] was devoted to the systamatising of the properties of semi-direct products. From a mathematical point of view, it was just because of this reliance that the representation theory of non-compact groups which are not semi-direct products required more general methods; motivated in this way, Bargmann [1947], Gel'fand and Naimark [1947], and Harish-Chandra [1947] all attacked the problem of the homogeneous Lorentz group and solved it completely. Again, Mackey [1949b] systematized this work also. This background illustrates the many obscurities in Wigner's work in 1939, quite apart from the relationships to the covariant wave equations.

The set of all unitary operators obtained in this way will form a ray representation of G, that is,

$$U_{g_1} U_{g_2} f = \omega(g_1, g_2) U_{g_1 g_2} f \qquad (3)$$

where ω is a multiplier of the representation (*cf.* 2.4.5)). Wigner was at pains to point out that knowing this representation is enough to determine the dynamics; that Eq.(3) replaces the equations of motion. Nevertheless he considered the question as to the existence of infinitesimal forms of Eq.(3). From Stone's theorem we know that for every 1-parameter abelian sub-group of G (which has a trivial multiplier group) one can define an unbounded essentially self-adjoint operator A such that $U_t = e^{-iAt/\hbar}$; in other words, there exists a dense domain of states D_A in $\mathcal{H}$ on which A is well-defined. Wigner now considered the question as to whether, for all the D_A's thus defined, the common intersection is dense or even non-empty. In fact he was able to show that for *irreducible* representations, if the latter, then also the former; that is, if only one state f in $\mathcal{H}$ is in the intersection D of all the domains D_A for all the one-parameter sub-groups of G, then $U_g f \in D$ for any $g \in G$, and since the set of states of the form $a_1 A_1 f + \ldots . a_i A_i f + \ldots$ is dense in $\mathcal{H}$, D must contain a dense set of states. If the representation is not irreducible, it is still possible to define D and the set D orthogonal to D. As subspaces (that is, after taking their closure) they decompose the representation (since if ϕ is orthogonal to f then it is orthogonal to $U_g f$, for any $g \in G$), into a "normal" and a "pathological" part. Wigner claimed that his subsequent analysis showed that the ILG has no pathological representations, thereby justifying *a posteori* the assumptions of Dirac and Majorana, who began from the outset with an analysis of the infinitesimal operators of a representation[5].

[5] There is no argument to this effect in the text that follows, and I can see no easy implication from this material either. The outline of a general theory of unbounded operators affilliated to factor representations is, however, already contained in Chapter XVI of Murray and von Neumann [1936], particularly lemma 16.4.3; Wigner's conclusion is a simple consequence of this lemma.

Section 4 is a review of the ILG in its classical form, containing well known results concerning its relationship to the universal covering group SL(2,C), the decomposition of a homogenous transformation into rotations and accelerations, and establishing that this group is simple.

Section 5 established that every projective representation of SL(2,C) is canonically generated by a unitary representation; in particular he showed that every multiplier of SL(2,C) is exact (that is, similar to the identity). This result was the first of its kind for a non-compact group, and was an important stimulus to the papers of Mackey [1958] and Bargman [1954]; the latter provided a systematic study of the multiplier group and was the first to determine the multiplier group of the Galilean group. The former subsumed the theory of projective representations under the imprimitivity theory through his construction of central extensions (see (3.1.5)). In this way the study of the multiplier group is reduced to the study of the cohomology of a certain class of central extensions. Wigner himself proceeded by using specific properties of the ILG to show that one can construct normalization factors for the unitary operators corresponding to the translations U_t and homogeneous transformations U_Λ so that any multiplier is removed, up to sign; this gave a representation "up to sign" of the ILG, a result that had already been obtained by Dirac using the Lie algebra of the ILG (Dirac [1935]; this result was cited by Wigner). In exact parallel to the rotation group, it follows that the universal covering group SL(2,C) has only exact multipliers, as likewise its semi-direct product with the translation group (the universal covering group of ILG). In this respect the fact that the HLG is simple (i.e. contains no proper subgroup invariant under the whole group) plays a crucial rôle; in the Galilean case, where the rotations and the boosts are invariant subgroups, this result fails, as was made clear by Bargmann [1954]. The homogeneous Galilean group is neither simple nor semi-simple; the boosts constitute an *abelian* invariant subgroup.

Section 6 contained the inducing construction, starting with a representation of the form Eq.(1) above restricted to a group orbit in $\hat{T}$; the basic idea has already been discussed. In justification for the assumption that the Hilbert space is of the form $L^2(\hat{T},\mathcal{K},\mu)$, and in introducing the cocycles $\phi(g,g^{-1}.p)$, he referred to the forthcoming von Neumann paper mentioned above. Von Neumann [1949] is entirely devoted to the decomposition theory of operator rings on a general class of (seperable) Hilbert spaces (the so-called *generalized direct sum*) of which this is a special case. All type I operator rings are there shown to decompose into countable direct sums of subrepresentations on spaces of the form $L^2(\hat{T},\mathcal{K},\mu)$, therefore also any commutative ring. Wigner assumed this form for $\mathcal{H}$ for the translation subgroup and argued from this to the representation of Eq.(1), with μ living on an orbit in $\hat{T}$[6]; it was in the clarification of these steps, together with the systematizing of the von Neumann decomposition theory (which made extensive use of measure theoretic properties of projection-valued measures) in application to l.c. groups, which led Mackey to the imprimitivity theory. In particular the notion of an operator ring on a Hilbert space which is a generalized direct sum is closely akin to the notion of an imprimitive representation of a group. We have already reviewed the decomposition theory (2.4.7); we need only recall that the semi-simple Lie groups and the commutative groups are all of type I (have only type I representations), and that the semi-direct product of two type I groups is type I, to conclude that the ILG is type I. This means that we have a canonical decomposition (i) into disjoint classes of rerpresentations, each of which (ii) is n copies of a multiplicity free representation, which is itself (iii) a direct sum or direct integral of disjoint irreducible

[6] From the physicist's point of view this is where the representation takes on an intuitive form; as a class of unitary operators acting on functions of momentum, restricted to the mass hyperbola. But why is this necessary? Wigner could not have supplied the answer to this question.

representations, in 1:1 correspondence with the inequivalent maximal Boolean sub-algebras of projectors in the commuting ring of the multiplicity free representation. This is in turn fixed by a measure class in a standard Borel space X; all multiplicity free representations arise in this way (see (2.4.7) for details). Most of these facts must have been known to von Neumann in 1937. The further crucial fact, that the measure classes are *all* provided by the group orbits, was not.

Section 7 contains the determination of all unitary representations of the stability sub-groups of the various orbits (with the exception of the orbits $m^2=0$, $p_0 = 0$, and $m^2<0$); from the inducing construction these then provide a classification for all unitary representations of the whole group. Section 8 is a discussion of the extended ILG (that is, its direct product with the discrete reflection group); we have already made mention of this material above. More details are included below (3.1.8).

For the rest of this section we shall apply the more general imprimitivity theory and decomposition theory to obtain the Wigner classification.

3.1.3. Semi-direct products.

In Section 2.4 we reviewed the imprimitivity theory, which for homogeneous systems of imprimitivity establishes a one-one correspondence, for each invariant measure class $\mathfrak{C}$ of a G-space X, between unitary equivalence classes of systems of imprimitivity and (G,X,M)-cohomology classes relative to $\mathfrak{C}$. Transitive systems of imprimitivity are a special class of homogeneous systems and for these we also reviewed the inducing construction, according to which for each transitive measure class (associated with the orbit $X = G.x_0$) there is a one-one correspondence between equivalence classes of systems of imprimitivity and equivalence classes of representations of the stability group L at x_0.

On the other hand we also considered the decomposition

theory according to which, for a type I representation of an arbitrary group, a unique direct sum decomposition into disjoint multiplicity-free parts is always possible, and for each multiplicity-free subrepresentation there exists a standard Borel space S such that the equivalence class of this subrepresentation is uniquely fixed by a Borel measure class in S.

The relationships between these theories are far-reaching and underlie many of the proofs of the theorems already cited; their connection is, however, particularly transparent in the case of semi-direct products, and for this case we also have a canonical construction of the G-spaces X and S.

The **semi-direct product** of a closed l.c. group K and a closed, normal *abelian* group T is defined whenever there exists a homomorphism of K into the group of automorphisms of T; that is, if for each $k \in K$ there is defined an automorphism $t \longrightarrow k(t) \in T$ of T. This being given, the semi-direct product group $G = K \overline{\times} T$ is defined as the set of all pairs (k,t) with the group product $(k_1,t_1)(k_2,t_2) = (k_1k_2, t_1k_1(t_2))$.

The subgroup $T\sim$ of G defined by the set of all pairs (e_K,t) (where e_K is the identity in K) is closed, normal and commutative and from the fact that $(k,t)(e_K,t')(k,t)^{-1} = (e_K, tk(a')a^{-1})$ we conclude that the inner automorphism of G induced by (k,e_T) coincides on T with the automorphism $(e_K,t) \longrightarrow (e_K,k(t))$. Identifying $T\sim$ and T, we conclude that k(t) can be written ktk^{-1}. The inhomogeneous Lorentz group is the connected part, homologous to the identity, of the semi-direct product of the translations group T (the additive group on $\mathbb{R}^4$) and the homogeneous Lorentz group K (the group of all invertible linear transformations of $\mathbb{R}^4$ which preserve the quadratic form $x_0^2-x_1^2-x_2^2-x_3^2$).

What we shall do is to establish that any irreducible representation L of the ILG implies the existence of an

invariant ergodic measure class $\mathfrak{C}$, and a multiplicity $n = \infty, 1, 2, \ldots$ in the dual space $\hat{T}$ (using the decomposition theory). But $\hat{T}$ is a G-space and, together with the invariant ergodic measure class $\mathfrak{C}$ and the multiplicity n, defines a homogeneous projection-valued measure on the Hilbert space $\mathcal{H} = L^2(\hat{T}, \mathcal{K}, \mu)$, where $\mathcal{K}$ is of dimensionality n and μ is any measure in the measure class $\mathfrak{C}$ (in particular it is ergodic, hence quasi-invariant). It then follows that this projection valued measure P, together with the representation H of the subgroup K (obtained by restricting L to K), must be a system of imprimitivity for K based on $\hat{T}$, which is moreover ergodic. But we know how to classify these from the imprimitivity theory (every ergodic system is homogeneous, *c.f.* Th.2.4.2 and the remarks preceding it). There is an important simplification at this point, that we actually only have to deal with transitive systems; the cohomology relative to transitive measure classes in $\hat{T}$ is completely known.

The first step is to apply the reduction theory; knowing that the ILG is type I, it is enough to consider the multiplicity free representations. Any multiplicity free representation defines a unique measure class in $\mathfrak{G}$, the space of all equivalence classes of irreducible representations. If we know the structure of $\mathfrak{G}$, the most general possible representation is the ILG is known. So we assume we have an irreducible unitary representation L of the ILG. If we consider its restriction L_T to the translation subgroup T it follows (Mackey [1949a]) that $L_T \simeq lM = M \oplus M \oplus \ldots \oplus M$ (l times) with $l = \infty, 1, 2, \ldots,$ where M is multiplicity-free.

The multiplicity free representations of a commutative group are, however, completely known. From Prontrjagin duality we know that $\hat{\hat{T}}$ and T are isomorphic which permits us to construct the canonical representation L^μ of T in $L^2(\hat{T}, \mu)$ defined by $L^\mu_g(f)(\chi) = \chi(g)f(\chi)$ (with μ a σ-finite Borel measure on $\hat{T}$ which is completely additive). L^μ is multiplicity free. From a result of Mackey [1949a] we also

know that every multiplicity free representation N of T is equivalent to some L^{μ}; moreover, L^{μ} and L^{ν} are unitarily equivalent if and only if μ and ν are in the same measure class, and disjoint if they are mutually singular. Therefore, we obtain the unitary equivalence classes of representations of T in one-one correspondence with the Borel measure classes of $\hat{T}$. More generally, for any unitary representaton N of T there is a *unique* P based on $\hat{T}$ such that $N_t = \int_{\hat{T}} \chi(t)dP(\chi)$ for all $t\in T$; we call it the projection valued measure **corresponding to** N. When N is multiplicity free, P is homogeneous.

The canonical action of K on $\hat{T}$ used in these theorems is the obvious one, provided by the uniqueness of that element y of $\hat{T}$ such that $y(t) = x(k^{-1}(t))$ for all $t \in T$ (the adjoint action of K on $\hat{T}$); y is therefore uniquely defined by each pair x,k and we can write $y = k(x)$, as the action of K on $\hat{T}$, which is therefore a G-space for K.

From the foregoing we see that any irreducible representation L of G determines a measure class $\mathcal{C}_L$ in $\hat{T}$ together with a multiplicity $l=\infty,1,2,\ldots$; but it does not follow that every measure class in $\hat{T}$ and any l can be obtained in this way. It was an important theorem of Mackey [1949b] that a measure class in $\hat{T}$ is of the form $\mathcal{C}_L$ for some irreducible representation L of G if and only if it is invariant and ergodic under K. But that means we can construct the associated homogenous projection valued measure on $\hat{T}$, and appeal directly to the imprimitivity theory, particularily Theorem 2.4.2. However, although this theorem enables us to construct representations (even irreducible ones) for an ergodic measure class $\mathcal{C}$ on $\hat{T}$, and classifies them according to the cohomolgy classes relative to $\mathcal{C}$, our knowledge of these cohomology classes is actually very limited. They are vastly more complicated than in the transitive case. The smooth orbit structure of the ILG now plays a crucial rôle in eliminating all such representations. One has the following theorem:

Theorem 3.1.1.

Let X be a standard G-space. Suppose that there exists a Borel set $D \subseteq X$ which intersects each orbit in exactly one point. Then, every ergodic invariant measure class on X is transitive.

(for proof see, *e.g.* Varadarajan [1970 Lemma 9.14]). Equivalently, this means that every ergodic invariant measure class is concentrated on some orbit (recall that a measure class $\mathcal{C}$ in X is transitive if there exists $x_0 \in X$ such that $X - G.x_0$ is a $\mathcal{C}$-null set). When every ergodic invariant measure class on X is transitive, X is said to have a **smooth orbit structure**. When X is $\hat{T}$ and G is the semi direct product of T and K, G is then said to be a **regular** semi-direct product. The anticedent is clearly true of the ILG when T is $\mathbb{R}^4$, $\hat{T}$ is $\mathbb{P}^4 \simeq \mathbb{R}^4$, and the union of the p_0 axis and the non-negative p_1 axis, for example, is Borel and intersects each orbit exactly once.

Theorem 3.1.1. therefore tells us the distinct orbits of K in $\hat{T}$ for each χ in $\hat{T}$ (the set $\mathcal{O}_\chi$ of all points in $\hat{T}$ to which χ can be mapped by an element of K) determine the unitary equivalence classes of the multiplicity free representation M occuring in the decomposition $L_T \simeq nM$.

Every irreducible representation L of G therefore determines a group orbit $\mathcal{O}_\chi$ in $\hat{T}$ and a multiplicity n, or equivalently a unique projection valued measure P corresponding to L_T based on $\hat{T}$ with multiplicity n, and whose measure class is concentrated on some orbit $\mathcal{O}_\chi$. It is an easy theorem (see, for example, Varadarajan [1970 Lemma 9.22]) that already the existence of a unique projection valued measure corresponding to the representation L_T is enough to ensure that L_K, P, is a system of imprimitivity for K based on $\hat{T}$. We state this theorem in full.

Theorem 3.1.2

Let N and H be representations of T and K respectively in a separable Hilbert space $\mathcal{H}$ and let P be the unique projection valued measure on $\hat{T}$ corresponding to N. Then there exists a unique representation L of $G = K \bar{\times} T$ such that $L_T = N$ and $L_K = H$ if and only if H,P is a system of imprimitivity for H based on $\hat{T}$.

We can now very simply apply the imprimitivity theory of Section 2.4, because for each irreducible L we know that the measure class of P corresponding to L_T is concentrated on an orbit in $\hat{T}$; therefore the system of imprimitivity defined, up to equivalence, by theorem 3.1.2, is transitive restricted to this orbit and we can apply theorem 2.4.2 directly. We thus obtain

Theorem 3.1.3.

Let L be an irreducible continuous unitary representation of the ILG on a separable Hilbert space $\mathcal{H}$. Then there exists an integer n, $n=\infty, 1, 2, \ldots$, and an orbit Θ_χ such that L is unitarily equivalent to the representation

$$(U_g f)(\chi) = r_g(g^{-1}.\chi)^{1/2}\phi(g, g^{-1}.\chi)f(g^{-1}.\chi)$$

acting on $L^2(\hat{T}, \mathcal{K}, \mu)$ where $\mathcal{K}$ is of dimension n, μ is a quasi-invariant measure in the invariant measure class $\mathcal{C}$ concentrated on Θ_χ, and ϕ is a $(G, \hat{T}, \mathcal{K})$-cocycle relative to $\mathcal{C}$. The orbit Θ_χ and the cohomology class of ϕ relative to $\mathcal{C}$ determine the unitary equivalence class of L completely.

Putting this together with the inducing construction of (2.4.8) we see that every irreducible representation is defined by two things; an orbit in $\hat{T}$ and an irreducible unitary representation of the stability subgroup of any point on this orbit on a separable Hilbert space $\mathcal{K}$. We summarize the various possibilities below. Going the other way we also know that any representation of the ILG can be uniquely decomposed as the direct sum of disjoint representations of the form nM, with M multiplicity free, and that each M can be uniquely written as the direct sum or direct integral of disjoint irreducible representations.

3.1.4. Momentum space

Above we have denoted the elements of $\hat{T}$ by the symbol χ, indicating that $\hat{T}$ is the set of characters of T. Since T is a vector group Pontrjagin duality extends to an isomorphism of T and $\hat{T}$ (the generalized Plancherel theorem) and we may use the language of the theory of Fourier transforms instead. There is in effect a canonical isomorphism between the character group of $T = \mathbb{R}^4$ and the vector space $\mathbb{P}^4$ of all real 4-vectors p_0, p_1, p_2, p_3 provided by[7] the map $p \longrightarrow \chi_p$, $\chi_p(x) = e^{ix.p/\hbar}$, where $x.p/\hbar$ is the canonical duality between $\mathbb{R}^4$ and $\mathbb{P}^4$ given by

$$p.x = (p_0x_0 - p_1x_1 - p_2x_2 - p_3x_3) \qquad (4)$$

so that $p.x$ is invariant when p transforms under the adjoint action of H on $\hat{T}$ and simultaneously x transforms under the canonical action of H on T.

Henceforward we shall refer to $\hat{T}$ as momentum space. The action of the translation subgroup L_T is given by $(L_t f)(p) = \chi_p(t)f(p) = e^{ip.t/\hbar}f(p)$. Note that physically $p = (E/c, \mathbf{p})$, where E is the energy and $\mathbf{p}$ the (3-vector) momentum.

3.1.5. Projective representations of the ILG.

There remains the question of multiplier representations; the result of Wigner [1939] shows that every multiplier representation of the ILG may be obtained as a unitary representation of the universal covering group of the ILG, the semi-direct product of the translation group with the group $SL(2,\mathbb{C})$. We shall henceforward denote this group K^* (with K, as before, the component of the homogeneous Lorentz group connected to the identity). K^* is semi-simple and hence of type I (Harish-Chandra [1953]). Since the semi-direct product is regular all of the foregoing may be applied to this group in place of K ; we need only note that the action of K^* on T is given by the

[7] L_t corresponds to the action $(x_0, \mathbf{x}) \longrightarrow (x_0 + t_0, \mathbf{x} + \mathbf{t})$. Note that the Hamiltonian generates the transformations L_{-t_0}.

covering homomorphism $\delta: K^* \to K$ which is a surjection onto the connected part of SO(3,1) with kernal $\{+1,-1\}$. That is to say, for each 2 x 2 complex matrix m of unit deter-minate $\delta(m)$ is that element in K which induces the same transformation on the vector space T as does the automorphism $\xi_t \to m\xi_t m^*$ acting on the vector space of Hermitian 2x2 matrices via the natural correspondence $\xi_t = t.\sigma$ (where σ is the 4-matrix given by $\sigma_0 = \mathbb{1}$, with σ_i the Pauli spin matrices[8] for $i = 1,2,3$).

It is clear that δ maps both the identity and minus the identity in K^* into the identity of K. but is locally isomorphic to and has the same Lie algebra as the homogenous Lorentz group. It is also clear that it has the same orbit structure in $\hat{T}$ so the only difference is that the cocycles which appear in the canonical representation of theorem 2.4.1 now define strict $(G^*, \hat{T}, \mathcal{K})$ - cocycles and the classification obtained rests on the class of all irreducible representations of the stability group of the covering group K^* rather than that of K (for a given orbit). Here G^* is the semi-direct product of K^* and T; in fact the cocycle dependence on the translations can be eliminated and we obtain the general irreducible representation for an element $(m,t) \in G^* \bar{x} T$ as follows:

$$(U_{(m,t)}f)(p) = r_m(m^{-1}.p)^{1/2} e^{ip.t/\hbar} \phi(m, \delta(m)^{-1}.p) f(\delta(m)^{-1}.p) \quad (5)$$

U acts on the Hilbert space $L^2(\Theta_p, \mathcal{K}, \mu)$ where μ is a quasi-invariant measure concentrated on Θ_p and ϕ is a strict cocyle with values in the unitary operators on the Hilbert space $\mathcal{K}$. In particular U is irreducible if and only if ϕ restricted to the stability group K_p^* of K^* at the point p defines an irreducible representation of K_p^* on $\mathcal{K}$.

[8] For an explicit representation see Eq.(61), (1.4.4).

The orbits in $\hat{T} = \mathbb{P}^4$ are:

$\Theta^+ = \{p: p.p = m^2c^2,\ m^2 \geq 0,\ p_0 > 0\}$

$\Theta_m^- = \{p: p.p = m^2c^2,\ m^2 \geq 0,\ p_0 < 0\}$

$\Theta_0 = \{p: p = 0\}$

$\Theta_m = \{p: p.p = -m^2c^2,\ m^2 > 0\}$

For $m^2>0$ the first two cases are, respectively, the (disconnected) positive and negative sheets of the mass hyperboloid; when $m^2 = 0$ we obtain three orbits, either Θ_0^+ or Θ_0^- (the past and future (disconnected) components of the light cone respectively) or the orbit Θ_0 containing the zero vector alone. The orbit Θ_m is the connected hyperboloid corresponding to spacelike momentum vectors. This orbit may be thought of as corresponding to imaginary mass; for $m^2>0$ we take m as the *positive* square root. Obviously at this stage the mass enters only as a convenient way of parametrizing the various orbits; in the Galilean case its rôle is somewhat different (see (3.2.2)).

An irreducible representation and its Hilbert space will be denoted correspondingly U_m^+ , U_m^- , $\mathcal{H}_m^+$, $\mathcal{H}_m^-$, etc. For the moment we observe that U^+ has the obvious interpretation as the representation corresponding to a positive energy particle and U^- that for a negative energy particle, with $\mathcal{H}_m^+$ and $\mathcal{H}_m^-$ the corresponding manifolds of states. There is an antiunitary equivalence between the two representations, so the associated Jordan * algebras are * isomorphic; their physical interpretation is elaborated in the following sections.

We obtain the irreducible representations for each orbit from the inducing construction. The stability group for each point $p \in \Theta$, is the subgroup K_p such that $k.p = p$ for $k \in K_p$. Suppose for $p' \in \Theta$ we have $k'.p' = p'$ (i.e. $k' \in K_{p'}$), with $g.p' = p$; then it is easy to see that $gk'g^{-1} \in K_p$ and that $K_{p'}$ and K_p are conjugate subgroups. Given a unitary representation U_K of K, U_{K_p} and $U_{K_{p'}}$ are unitarily equivalent; since the representation of G^* for given Θ is fixed by the unitary equivalence class of the

representations restricted to the stability group of a point p in Θ, we see that it does not matter which point we pick in a given orbit. Choosing the point p = (+mc,0,0,0) in the orbit Θ_m^+, the stability group, as a subgroup of K (the connected part of SO(3,1)), is clearly the orthogonal group SO(3) (the condition that $\Lambda^t G \Lambda = G$, where Λ is a 4x4 real invertible matrix and G has diagonal form +1,-1,-1,-1, reduces to the condition for orthogonality). In classical terms, this choice of momentum vector is equivalent to transforming to a co-ordinate system in which a particle is at rest (i.e. with time-like component only); this vector is invariant under any spatial rotation. As a subgroup of K^* (that is, the subgroup of all $k \in K^*$ such that $\delta(k).p = p$) it is the unitary subgroup SU(2) of $SL(2,\mathbb{C})$, the universal covering group of SO(3). The irreducible unitary representations of this group are well known; they are the 2j+1 dimensional matrix representations of the rotation group, j=0,1/2,1,3/2..., which we shall denote D_j; only integral j irreducible unitary representations of SO(3) exist. We shall denote the associated (induced) representations $U_m^{\pm,j}$.

The general form of these representations can then be written as follows: if $\mathcal{K}^j$ is the space of the representation D^j (therefore isomorphic to $\mathbb{C}^{2j+1}$) and ϕ is a strict $(K^*, \Theta_m^{\pm}, U)$ cocycle with values in the unitary group U of $\mathcal{K}^j$, such that ϕ (denote ϕ^j)defines the representation D_j at $(\pm mc,0,0,0)$, then $U_m^{\pm,j}$ acts in the Hilbert space $\mathcal{H}_m^{\pm,j} = L^2(\Theta_m^{\pm}, \mathcal{K}^j, \mu)$ (with μ a quasi-invariant measure which lives on $\Theta_m^{\pm}$) and has the canonical form of Eq.(5) with $\phi = \phi^j$.

Since we are not in this thesis concerned with the zero-mass case I shall not consider the remaining orbits. I refer to Varadarajan [1970] for a systematic account. Although the inversions will be of some relevance in what follows we shall only need explicit formulae in the scalar case; these are simple to derive. In the general case, we only need to know that the space inversion operator is unitary, the time inversion antiunitary, and that they do *not* interchange the mass hyperbolae. There is a canonical map between the Wigner

representations (for given mass and spin) on each hyperboloid. This map is *antiunitary*; in the scalar case it is given by:

$$\mathfrak{C}: \mathcal{H}^{+}_{m} \longrightarrow \mathcal{H}^{-}_{m}, \quad (\mathfrak{C}f)(p) = \overline{f(-p_0,-p_1,-p_2,-p_3)} \tag{6}$$

and (up to equivalence) it may be identified with the 1-particle charge conjugation operator[9]. The parity and time inversions (scalar case) are:

$$\begin{aligned} \mathfrak{P}&: \mathcal{H}^{+}_{m} \longrightarrow \mathcal{H}^{+}_{m}, \quad (\mathfrak{P}f)(p) = f(p_0,-p_1,-p_2,-p_3) \\ \mathfrak{T}&: \mathcal{H}^{+}_{m} \longrightarrow \mathcal{H}^{+}_{m}, \quad (\mathfrak{T}f)(p) = \overline{f(p_0,-p_1,-p_2,-p_3)} \end{aligned} \tag{7}$$

With this we conclude the analysis of Hilbert space representations of the Lorentz group. We consider the irreducible representations to define the theory of *relativistic quantum mechanics*[10]. The decomposition theory shows that *any* Lorentz invariant QFT on Hilbert space, on which a unitary (and highly reducible) representation of the ILG is defined, can be interpreted atomistically. Note that the failure of the von Neumann uniqueness theorem for such systems has nothing to do with the representations of the ILG, but rather with the CCR's for the field. Only for irreducible representations may the CCR's be directly associated with the Lie algebra of the ILG.

[9] Its covariant Fourier transform then transforms simply as: $\mathfrak{C}: f(x) \longrightarrow \overline{f(x)}$, as may easily be verified.

[10] As we shall see when we establish a Born interpretation, we have everything in NRQM - except a general theory of interactions. It is a principle goal of Sections 3.2-3.4 to understand why this is so. Here we note that there are difficulties in formulating an interacting theory even in classical theory (Currie et al [1963]. This theorem has been circumvented in various ways (see, e.g. Pons [1983] and references therein), and is not I believe of fundamental relevance.

3.2 The Relativistic Wave Equations

First quantization is a mystery.....

E. Nelson

3.2.1 Introduction.

The Mackey theory in application to the Lorentz and Galilei groups represents the most extraordinary achievement: there are essentially only two parameters $m \in \mathbb{R}$, $2s \in \mathbb{Z}$ which uniquely define *all* the irreducible unitary Hilbert space representations, and any (unitary) representation whatsoever can be obtained as a direct sum over these irreducible representations. If one is to describe a physical system using a complex Hilbert space, and this description is to be independent of the inertial co-ordinate systems, then these must be the essential building blocks of the description, even in field theory.

I believe this point cannot be stressed too strongly. It does not stretch the point too far to say that a Minkowski or Galilean world, described in Hilbert space, must be an atomistic world; from this result alone we are led to believe that *any* quantum field theory (for example) must admit a particle interpretation[1].

By and large, the physics community has remained curiously indifferent to the wider implications of the Mackey theory. From a philosophical viewpoint, however, and especially if one is prepared to acknowledge the abstract characterization

[1] Although it need not be of Fock type; that is, the representation may have infinite multiplicity. See Section 3.5 for a discussion of non-Fock representations. Also the continuous spin representations cannot be excluded a priori; their atomistic interpretation is obscure.

of Hilbert space theory, the implications of the essential uniqueness of an elementary description of the world are little short of revolutionary. There may be other ways in which the notion of "most simple parts" may be formulated; but if the world is Minkowski or Galilean, *this* characterization of elementarity will remain a mathematical fact of all possible Hilbert space descriptions. With a slight shift in emphasis, one might also say that atomism, as a characteristic of any Hilbert space description of the world, stands or *may* fall on the nature of physical spacetime.

In the last half-century a tremendous gap has opened between the physics and mathematics communities; this is nowhere more apparent than in the response of these communities to the Wigner paper. The mathematics community was prodded into a sustained attack on the general representation theory of non-compact groups, but paid no attention to the covariant representations used in physics. The physics community continued to make use of the covariant wave equations with perfunctory reference to the fact that Wigner had classified all irreducible representations.

In this section we review, not the systematic exploration of the relationship between the Wigner and covariant representations, but the fragmentary insights yielded by the fortuitous discovery of a formal correspondence: the Foldy-Wouthuysen transformation. We also review the elaboration of the notion of *particle localization*; the two together establish a formal correspondence between the Wigner representation for $m > 0$, $s = 1/2$, and the Dirac equation.

For the physicist, the problem of locality remained (*circa* 1934) the only outstanding difficulty of free field theory: unlike conceptual problems concerning the relationship of relativistic to non-relativistic theory, locality plays an essential rôle in defining a correspondence between theory and experiment. Nevertheless one can view the difficulties of defining a Born interpretation for particle position with

varying degrees of equanimity. To some extent one can regard predictions of the theory, which concern the momenta of particles in a scattering situation, as adequate to the task of providing empirically testable statements. To some extent it is a matter of taste as to when one will consider a theoretical proposition as empirically meaningful: one has to draw the line somewhere, and no quantum theory will describe actual laboratory procedures. In this spirit we have already encountered one possible response which makes light of the absence of a Born interpretation (1.4.5): it is in any case not possible to experimentally localize a particle within a region small in comparison to its Compton wavelength, because the uncertainty in energy will exceed the threshold for pair creation.

To repeat the objection to this line of argument made earlier, we cannot even know that a particle is approximately localized unless we know what it means to say that a particle is precisely localized. And against the adequacy of a Born interpretation based on the measurement of momentum, I believe there is a significant difference between the association of the position of silver grains in photographic emulsion with particle position, and the curvature of ionization trails with particle momenta[2]. For these reasons the existence of a Born interpretation for particle position is more than of passing interest; it is, in an older terminology, a matter of freeing the relativistic quantum theory from a dependence on the correspondence principle.

[2] I refer to (1.4.5) for details.

3.2.2. Comparison of the Galilean and Lorentz groups; position operators.

In NRQM the position operators play a fundamental rôle in the *mathematical* definition of the theory, through their appearance in the CCR's. How is it that they do not appear in the CCR's for the relativistic theory? To answer this question, we shall consider the representation theory of the *Euclidean group* $\mathcal{E}$ (comprising the space translations and rotations).

We shall begin with the following situation: we have an N particle system with n=3N and a configuration space $\mathbb{R}^n$; we suppose that we have a (weakly continuous) unitary representation U of $\mathcal{E}$ on a complex Hilbert space $\mathcal{H}$, and a projection-valued measure P on $\mathcal{H}$ which satisfies the imprimitivity relationship and which is based on configurations space. That is, we have a system of imprimitivity (U,P) based on $\mathbb{R}^n$ such that $U_g P_E U_g^{-1} = P_{g.E}$ (E Borel in $\mathbb{R}^n$), where g.E is the configuration space action of $\mathcal{E}$ on $\mathbb{R}^n$ (as opposed to the adjoint action on the dual space familiar from Section 2.4). From the projection-valued measure we can construct a homomorphism from $\mathcal{C}^\infty(\mathbb{R}^n)$ into the set of all operators on $\mathcal{H}$, with real functions corresponding to self-adjoint operators, in the standard way: $A_f = \int f(x)dP(x)$, $f\in\mathcal{C}(\mathbb{R}^n)$. This is the algebra of functions of a self-adjoint operator. We may think of it as the set of all functions of the position operators.

The unitary representation of the Lie group $\mathcal{E}$ will, however, lead to another class of self-adjoint operators, namely those which generate unitary representations of the one-parameter sub-groups of $\mathcal{E}$, via Stone's theorem. So we suppose we have such an operator B_X, and the corresponding one-parameter group $U_X(s) = e^{(-iB_X s/\hbar)}$; since now P obeys the imprimitivity relationship, the A_f's will transform as $U_g A_f U_g^{-1} = A_{g.f}$ where $(g.f)(x) = f(g^{-1}.x)$, and from their transformation under U_X we can work out the commutation relationships between B_X and the A_f's, because U_X will determine an action on the f's .

This is as follows; the one-parameter groups in $\mathcal{E}$ are in one-one correspondence with any independent set in the Lie algebra $\mathfrak{g}$; each X in $\mathfrak{g}$ will define (via the adjoint action) an action on $\mathcal{E}$ and hence on $\mathbb{R}^n$ (the exp action). That is for each s, $x \longrightarrow \exp(sX).x$ is a diffeomorphism on $\mathbb{R}^n$. The vector field τ_X of this diffeomorphism by definition has the action $(\tau_X f)(x) = \frac{\partial}{\partial s}\,(f(\exp(sX).x)\big|_{s=0}$. So for each $X \in \mathfrak{g}$ there is an action on $\mathcal{C}(\mathbb{R}^n)$ and hence on the algebra of the A_f's. From this we can deduce the following commutation relationships[3]:

$$[A_f, A_g] = 0 \qquad \text{(i)}$$

$$[A_f, B_X] = i\hbar A_{\tau_X f} \qquad \text{(ii)}$$

$$[B_X, B_Y] = i\hbar B_{[X,Y]} \qquad \text{(iii)}$$

$$aB_X + bB_Y = B_{aX+bY} \qquad \text{(iv)}$$

We note that (i) and (iv) are familiar poperties in connection with commutation relationships among the 3n position operators or the relationships among the 3n momentum operators. In fact when f is the map $\iota_i : (x_1, .., x_{3n}) \longrightarrow x_i$ then A_{ι_i} is the i^{th} position operator of the system.

Now suppose that B_X generates the translations U_g, with g in the one-parameter subgroup of translations in the j^{th} co-ordinate direction; we shall write the corresponding element of the Lie algebra as p_j. The relationship (ii) then takes the form, for B_{p_j} and A_{ι_i}, of the CCR $[A_{\iota_i}, B_{p_j}] = i\hbar A_{\tau_{p_j}\iota_i} = i\hbar E_{ij}$, where E_{ij} is the projection operator defined by $\delta_{ij}\int dP$. This is the usual CCR between position and momentum operators.

It would be nice to extend this analysis to a space-time symmetry group and obtain the CCR's for an n-particle system, thus making the connection between quantum mechanics

[3] Observe that the imagninary number which enters (2) and (3) is the canonical complex structure on H (that is, it defines what we mean when we say that H is a complex Hilbert space). This will be of importance in Section 3.4.

and its space-time group very clear and direct. As we indicated in (2.4.6), this cannot be done in any straightforward way, with a 3N dimensional configuration space as base. Trying to use N copies of Minkowski space as base leads to an altogether different notion of locality: a system localised in space *and* time. The more natural system of imprimitivity based on the dual space (momentum space) leads, however, to (functions of) the energy-momentum operators defined via $A_f = \int f(p)dP(p)$. If now the position operators arise as generators of some set of one-parameter subgroups of G, then since the momentum operators are themselves elements of the Lie algebra of G, the only CCR's obtained in this way between position and momentum will come from the group structure itself, via (iii) above; that is, from the Lie brackets of $\mathfrak{g}$.

However this strategy is not helpful in the case of the Lorentz group; there is no commuting set in the Lie algebra with the right Lie brackets with the generators of translations. For the Galilean group, however, there is such a commuting set (for the space-translation generators, the momenta), which generate the Galilean boosts. However they have *vanishing* Lie brackets with the momenta. This apparent contradiction is resolved by the observation that there are *projective* representations of the Galilean group with non-exact multipliers, which are not equivalent to any true representation (not even of the universal covering group). When this happens there is a general theory which tells one how to construct a new Lie algebra, the *central extension* of $\mathfrak{g}$, of which the unitary representations are the projective representations of the original algebra. Essentially the central extension adjoins to $\mathfrak{g}$ certain "neutral elements" (that is, elements in $\mathfrak{g}$ which have vanishing Lie bracket with every other element in $\mathfrak{g}$). These neutral elements appear in Poisson bracket representations of the group in the context of classical theory, and in the existence of "multipliers" (that is, phase factors) in Hilbert space representations (projective representations) in quantum theory. In the present context just such a neutral element arises for the Lie bracket of the generators for Galilean

boosts b_i and those for translations p_j; it is the *mass* of the representation (denote m). One then has the Lie bracket $[b_i, p_j] = m\delta_{ij}$ (which carries over to the usual CCR between position and momentum via (iii); b_i/m corresponds to the position operator). Note that the mass associated with a representation in the Lorentz case derives from a different idea altogether, namely that of the *Casimir invariants* of the Lie algebra. Of course, once one has the central extension of the inhomogeneous Galilean group, one can determine its Casimir invariants and the neutral elements in the Lie algebra will appear then as Casimir invariants of the central extension of $\mathfrak{g}$. But there is a difference; fixing a value to a neutral element in $\mathfrak{g}$ determines the representation. The *irreducible* representations are then defined by the values of the *remaining* Casimir invariants. In the Lorentz case, reducible representations are possible in which the mass is not a constant (although it remains an invariant function of the appropriate elements in the Lie algebra). This is not possible in the Galilean theory; this fact has an intimate connection with the Bargmann superselection rule (Bargman [1954]; see also Sudarshan and Mukunda [1974] for general background).

In (2.5.6) we obtained a different group-theoretic interpretations of the CCR's in their global form; we considered the Weyl algebra as a system of imprimitivity for the Euclidean[4] group $\mathcal{E}$ based on $\mathbb{R}^n$. We now see that there is another way of looking at the CCR's: as the infinitesimal form of a projective representation of the inhomogeneous Galilean group based on $\mathbb{P}^4$ (where the Weyl algebra is the restriction of this representation to the two normal abelian subgroups, the space translations and the boosts)[5]. Here the full Galilean group is employed, and the position operators

[4] We actually considered only the translation sub-group. The above is the natural generalization to include rotations and the global form of the CCR's for angular momentum.

[5] This is related to the formulation of the Weyl algebra given in (2.4.5): as a projective representation of the additive group on the cotangent bundle of a linear classical system (2.5.7).

thus defined are fully covariant (they satisfy the intuitively correct Lie brackets with all the elements in the Lie algebra of the central extension). This formulation is inapplicable to the Lorentz case (the pure Lorentz transformations or boosts forms a sub-group which is neither normal nor abelian). Applying the first formulation to the full Lorentz group with spacetime as base leads to space-time locality, and

it is not possible to define a system of imprimitivity for the whole group based on $\mathbb{R}^3$; therefore one can only obtain CCR's which involve configuration space observables for a subgroup of the inhomogeneous Lorentz group, and the largest such subgroup is the Euclidean group. Whether this can be done in a way consistent with the representation of the entire group defined with $\mathbb{P}^4$ as base is another question; see (3.2.9) below. If it *is* possible, we may expect that the configuration space observables thus defined will *not* transform covariantly under the full group (because the imprimitivity relationship for the projection-valued measure which defines the position observables will not be satisfied for the whole group); that is, we expect that the relationship $U_g P_E U_g^{-1} = P_{g.E}$ will *not* be satisfied when g does not lie in the Euclidean subgroup (in particular when g is a boost). The fact that in the Galilean case one can actually obtain the position operators as (constant multiples of) the generators of the boosts, is a remarkable and apparently fortuitious circumstance which leads to a fully covariant notion of localization for this group.

3.2.3. *Résumé*; comparison of the Schrödinger and relativistic wave equations.

By 1934 it was widely acknowledged that there are severe difficulties in formulating a Born interpretation for the relativistic wave equations. We recall (1.4.5) that a number of thought experiments had been proposed to show that the concept of locality cannot be applied to length scales shorter than the Compton wavelength. At the same time the formal difficulties of defining a position operator were evident. For the free KG equation

$$(\Box + m^2c^2/\hbar^2)\varphi(\mathbf{x},t) = 0 \qquad (1)$$

one forms the vector space $\mathcal{V}$ of *positive* frequency solutions; the invariant sesquilinear form

$$(\varphi,\psi) = i\hbar \int(\bar{\varphi}\,\dot{\psi} - \dot{\bar{\varphi}}\,\psi)d^3x \qquad (2)$$

(where $\dot{\varphi} = \frac{\partial}{\partial t}\varphi$) is then positive definite and hence defines an invariant inner product on $\mathcal{V}$. Therefore $\mathcal{V}$ is pre-Hilbert and we denote its completion $\mathfrak{h}^+$, the *Hilbert space of positive frequency solutions* of the KG equation. Similarily, we can define the Hilbert space $\mathfrak{h}^-$ of negative frequency solutions (but note that to obtain a positive definite inner product *we must reverse the sign of the expression* Eq.(2)).

This procedure differs markedly from the non-relativistic case, where one has a Hilbert space of unconstrained functions on $\mathbb{R}^3$ (apart from the mild L^2 integrability condition). It is a principal objective of this section to understand why this is so (see also (3.4.8)). One appears to avoid the problem by going over to momentum space: one has the integral representation (*cf.* Eq.(3) (1.4.2)):

$$\varphi(x) = 2^{1/2}(2\pi\hbar)^{-3/2}\int e^{-ip.x/\hbar}\delta(p^2 - m^2c^2)\Theta(p_0)f(p)\,d^4p$$

(where $\Theta(x) = 1$ for $x > 0$ and zero otherwise) which, for (almost) arbitrary f on $\mathbb{P}^4$, clearly defines a positive frequency solutions of Eq.(1); integrating over p_0 we obtain[6]

[6] Whenever we have a function f on $\mathbb{P}^4$ with values f(p) we mean by f(**p**) the values of the function g on $\mathbb{P}^3$ defined by $g(y) = f((y^2 + m^2c^2)^{1/2}, y_1, y_2, y_3)$, where $y^2 = y_1^2+y_2^2+y_3^2$. Where

$$\varphi(x) = (2\pi\hbar)^{-3/2}\int_{\mathbb{P}^3} e^{-ip.x/\hbar} f(\mathbf{p})d^3p/\sqrt{2p_0}. \qquad (3)$$

Using this transform (the covariant Fourier transform) one can verify directly that the inner product takes the form:

$$(f,g) = \int_{\mathbb{P}^3} \overline{f(\mathbf{p})}g(\mathbf{p})d^3p/p_0 \qquad (4)$$

(where $p_0 = (\mathbf{p}^2+m^2c^2)^{1/2}$). That is, $\mathfrak{h}^+ \simeq L^2(\mathcal{e}^+,\mu^+)$ where $\mathcal{e}^+$ is the positive mass hyperboloid and $d\mu^+ = d^3p/p_0$ on $\mathbb{P}^3$. It is a theorem (see, *e.g.* Varadarajan [1970 Th.12.2]) that μ^+ is the *unique* invariant measure on $\mathcal{e}^+$ (up to a constant multiple); the space $L^2(\mathcal{e}^+,\mu^+)$ is therefore unitarily equivalent to $\mathcal{H}_m^+$ as defined in the previous section. We can now regard the $f(\mathbf{p})$'s as the relativistic analogues of the momentum space time-zero wave-functions, constrained only by an L^2 integrability condition.

The correspondence between functions f on $\mathbb{P}^4$ and functions φ on $\mathbb{R}^4$ obtained in this way can also be defined by the procedure: for each f on $\mathbb{P}^4$, define the (tempered) distribution $T_f = 2^{1/2}(2\pi\hbar)^{1/2}\delta(p_0^2-mc^2)\Theta(p_0)f(p)$, and then take its usual (inverse) Fourier transform $\check{T}_f$. It can be shown that $\check{T}_f$ is a tempered distribution which is a positive frequency solution to the KG equation.

The invariance of μ ensures the invariance of Eq.(2) under Lorentz transformations; that is, under a transformation (L,y): $x \to Lx+y$ of the inhomogeneous Lorentz group, we suppose φ transforms as $\varphi \longrightarrow \varphi'$, where $\varphi'(x) = \varphi(L^{-1}(x-y))$. It is apparent that under the transform Eq.(3) this transformation of φ corresponds to the transformation $f(p) \longrightarrow e^{ip.y/\hbar}f(L^{-1}p)$; this transformation is moreover unitary on $\mathcal{H}_m^+$ so that the positive frequency solutions of Eq.(1) correspond under Eq.(4) to the unitary representation $U_m^{+,0}$ on $\mathcal{H}_m^+$ studied in (3.1.6).

It is clear that if $\varphi\in\mathfrak{h}^+$, then the function $x^1\varphi(\mathbf{x},t)$ does not necessarily lie in $\mathfrak{h}^+$; indeed, it may not even lie in

we begin from the outset with a function f on a three dimensional space we shall usually write the variable in bold type to emphasise this fact. No confusion will ensue.

$\mathfrak{h}^+ \cup \mathfrak{h}^-$ (that is, $x^1\varphi(\mathbf{x},t)$ may not satisfy the KG equation at all). One can of course define a position operator x^1_{op} on $\mathfrak{h}^+ \cup \mathfrak{h}^-$ by taking $x^1_{op}\varphi$ as that solution of the KG equation which has Cauchy data $x^1\varphi(\mathbf{x},0)$, $x^1\dot{\varphi}(\mathbf{x},0)$; one must then determine whether this defines a linear operator on $\mathfrak{h}^+$. It is easiest to exploit the correspondence between $\mathfrak{h}^+$ and $\mathcal{H}^+ = L^2(\Theta^+,\mu^+)$ provided by the transform Eq.(3); it is then apparent that

$$(\varphi, x^1_{op}\psi) = \int_{\Theta^+} \bar{T}_\varphi \left(\frac{\partial}{\partial p_1} T_\psi\right) d\mu^+ \neq \int_{\Theta^+} \left(\frac{\partial}{\partial p_1} \bar{T}_\varphi\right) T_\psi \, d\mu^+$$

(because of the p dependence in the measure μ^+), where

$$\hat{T}_\varphi(p) = (2\pi\hbar)^{-2} \int e^{ip.x/\hbar} T_\varphi(x) d^4x$$

Therefore x^1_{op} is *not* self-adjoint on $\mathfrak{h}^+$ (or $\mathcal{H}^+$).

For the Dirac equation, on the other hand, one finds again that multiplication by x^1 is not a satisfactory position operator, but for somewhat different reasons. To see this, we write the Dirac equation

$$(i\hbar\gamma^\mu\partial_\mu - mc)\psi(\mathbf{x},t) = 0$$

where $\psi: \mathbb{R}^4 \longrightarrow \mathbb{C}^4$ is a Dirac spinor, and the γ^μ are complex 4×4 matrices which satisfy

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = [\gamma^\mu,\gamma^\nu]_+ = 2g^{\mu\nu},$$

in the form

$$c(-i\hbar\gamma^0\gamma^1\partial_1 + \gamma^0 mc)\psi = i\hbar \frac{\partial}{\partial t}\psi \qquad (5)$$

where $-i\hbar c\gamma^0\gamma^1\partial_1 + \gamma^0 mc^2$ is the Dirac Hamiltonian. As before we define the Hilbert space $\mathfrak{h}^+$ of positive frequency solutions, complete in the invariant inner product

$$(\varphi,\psi) = \int \Sigma \overline{\varphi(x)}\psi(x) d\, x^3 \qquad (6)$$

(the summation is over the bi-spinor components). A difference now appears from the scalar case; not all the components of a positive frequency solution can be independent. This is obviously the case, for were it not, it would not be possible to obtain *any* negative energy solutions of Eq.(5) (since the time dependence is uniquely determined by $\varphi(x,0)$). That is, although as initial data we may choose an arbitrary differentiable function $f: \mathbb{R}^3 \longrightarrow \mathbb{C}^4$ and define $\varphi \in \mathfrak{h}^+ \cup \mathfrak{h}^-$ by $\varphi(\mathbf{x},0) = f$, $\dot{\varphi}(\mathbf{x},0) = -\frac{i}{\hbar} Hf$, we cannot in this way obtain a correspondence between f and φ in $\mathfrak{h}^+$; as a consequence we cannot define a linear operator

on $\mathfrak{h}^+$ by lifting the action of multiplication by x^1 on f.

Pursuing this point will take us back to the analysis of (1.4.5) and Schrödinger's distinction between "even" and "odd" operators; the necessary constraint on those $f:\mathbb{R}^3 \longrightarrow \mathbb{C}^4$ which give rise to $\varphi \in \mathfrak{h}^+$ is not obeyed by $f' = x^1 f(\mathbf{x})$ (that is, φ' defined by f' has a negative frequency part). We shall shortly exhibit this analysis in the course of defining the Foldy-Wouthuysen representation.

There is, however, a simple observation which makes it clear that in a fundamental sense the spatial co-ordinates (configuration space) used in this representation (that is, when one solves the KG equation for positive frequencies) *cannot* refer to particle localization. This is because such solutions (which are well-behaved as functions on momentum space) must be entire functions.

There is an elementary demonstration of this fact[7]; for f(p) bounded and of compact support[8], then from Eq.(3):

$$\varphi^+(x) = (2\pi)^{-3/2}\int_{\mathbb{P}^3} e^{-ip_0.t/\hbar} e^{i\mathbf{p}.\mathbf{x}/\hbar} f(p)\, d^3p/\sqrt{2p_0}$$

it follows that as a function of t, φ^+ is the boundary value of an entire function of t (with t complex on the lower half plane; the exponent is then negative definite since p_0 is positive definite); hence $\varphi^+(t)$ is entire and therefore it cannot vanish on any open set (in time) unless it vanishes at all times. So much is also true of the non-relativistic theory, and we have one more way of understanding why there is no meaningful concept of localization in *time* (no such notion can be formulated within the manifold of *positive energy* solutions).

The difference between the Galilean and Lorentz theories arises when one considers the implications of this result

[7] I am indebted to Professor R.Streater for this argument. See also Streater [1988 p.138].

[8] Or for every f which is infinitely differentiable and of compact support; it is a well-known theorem (see, e.g. Gel'fand and Shilov [1958 II,1.1]) that the Fourier transform of any such f is entire.

for a positive frequency solutions which vanishes over an open set in *space*. The KG equation is hyperbolic, the Schrödinger equation parabolic; in the first case, but not the second, the value of a solution at a point $(\mathbf{x},t)$ is determined by its values on the intersection of a spacelike hypersurface with the past light cone from the point $(\mathbf{x},t)$ (a Cauchy surface, in the terminology of (2.5.3)). That is, $\varphi^+(\mathbf{x},0)$ and $\frac{\partial}{\partial t}\ \varphi^+(\mathbf{x},t)\big|_{t=0} = \dot{\varphi}^+(\mathbf{x},0)$ with $\mathbf{x} \in \mathcal{S}$ (see diagram) determines the value of φ^+ everywhere within the future (and past) development $\mathcal{S}^{\pm}$ (region shaded; that is, φ^+ and $\dot{\varphi}^+$ on $\mathcal{S}$ are Cauchy data for $\mathcal{S}^{\pm}$). If, then, $(\varphi^+,\dot{\varphi}^+)$ vanish on $\mathcal{S}$ they vanish everywhere in $\mathcal{S}^+ \cup \mathcal{S}^-$ and in particular φ^+ vanishes on an open set in time; therefore it vanishes at all times.

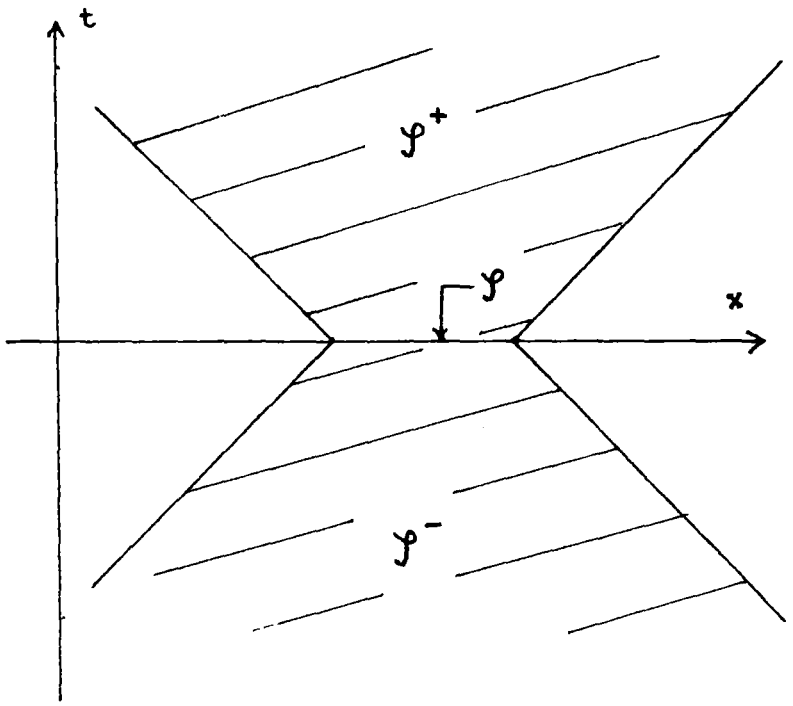


Fig 3.2.1

This is one occasion where the figurative use of mathematics (in particular the use of the Dirac δ function) has led to real conceptual error. It is not just that $\bar{\varphi}\overleftrightarrow{\partial_t}\varphi$ cannot have the meaning of the position probability density (a fact that was reasonably well-understood from the 30's onwards); the time-zero configuration space positive frequency solutions of the KG equation cannot play anything like the rôle of

Cauchy data, and is in this sense quite unlike the configuration space wave-function in the non-relativistic case. Only the momentum space states are unconstrained.

3.2.4. Newton-Wigner localization.

Throughout the 30's and 40's physicists appeared disposed to forgo a Born interpretation[9]; during this same period, increasing doubt was thrown on the utility of Lagrangian field theory in hadron physics because of the inapplicability of perturbartion theory. From the 40's onwards, the growing popularity of the S-matrix theory made possible a general philosophy in which the spacetime description of dynamical processes was increasingly viewed as irrelevant or even meaningless. The most explicit accounts of this philosophy were published much later:

> In the conservation postulate the notion of translational invariance appears for the first and only time. One may completely avoid the introduction of space-time coordinates by simply taking energy-momentum conservation to be exactly the requirement of translational invariance. However, it is apparent that one could perform a formal Fourier transformation on the momentum-energy variables, introducing thereby formal space-time coordinates. Formal translation invariance is then equivalent to energy-momentum conservation. (Stapp [1962 p.2141-2])

That such ideas were influential much earlier is indicated by the defensive remarks at the close of Newton and Wigner's successful construction of a position operator (and associated generalized eigenstates):

> One may wonder, even in the case of elementary particles,

[9] Although many indulged in wishful thinking; that is, physicists continued to interpret the formalism, when written in configuration space coordinates, as if a Born interpretation was applicable. This has survived in a number of standard texts used to this day; see, e.g. Bjorken and Drell [1965] in their discussion of the interpretation of propagators (e.g. p.63). On the whole these inaccuracies were innocuous; somewhat more surprising is the Bloch interpretation of the multiple time formalism (Bloch [1934]), where, in explicitly addressing the question of interpretation, one might expect a sharper focus on this issue. Bloch made no reference to the difficulties of defining a Born interpretation for the Dirac theory.

> whether the determination of the localized states and position operators has much significance. Such doubts might arise particularly strongly if one is inclined to consider the collision matrix as the future form of the theory. One must not forget, however, that the customary exposition of the theory refers only to questions about cross-sections. There is another interesting set of questions referring to the position of the scattered particles: how much further back (*i.e.* closer to the scattering center) are they than if they had gone straight to the scattering centre and then continued in the new direction without any delay. In order to answer such questions in the relativistic region, one will need some definition of localised states for elementary systems. From this point of view it is satisfactory that the localized states could be defined without ambiguity just for these systems. (Newton and Wigner [1949]).

Neither were they motivated by a concern to define a Born interpretation in the relativistic case; Newton had investigated the representations of the de Sitter group for his PhD dissertation, and it was only to assist the physical interpretation of these representations that he, with the assistance of Wigner, considered the possibility of defining position operators "on an invariant theoretical basis"; as a preliminary to that, they first considered this problem in flat spacetime.

But the paper of Newton and Wigner did not appear altogether without precursors; the preceding year Pryce had developed the *classical* relativistic theory of the centre of mass of a collection of free point particles (Pryce [1948]). It is remarkable that so little progress had been made with this problem since the inception of relativity theory; the work of Focker ([1929]), and Born and Fuchs ([1940]), having proved unsatisfactory in various respects. In particular the naïve approach, of defining the weighted means of the (rectilinear) coordinates of the individual particles (with weights given by the rest masses), yields a centre of mass which is not in general at rest in a frame in which the total momentum is zero, nor is it independent of the inertial frame with respect to which the coordinates of the particles are defined.

Pryce derived an expression for the centre of mass, the components of which have vanishing Poisson brackets, which

satisfies the intuitive requirements of this notion and which was defined in terms of integrals of the energy-momentum tensor. As such it was possible for him to take this expression over to the Dirac and integral spin theories (symmetrizing expressions in which an ambiguity in the order of the generators of the Lorentz group enter into the expression for the energy-momentum tensor). Similar ideas were independently proposed by Finkelstein [1948] and Moller [1949].

Newton and Wigner proceeded from a more foundational point of view. They did not, however, use the Wigner representation, but rather the momentum space covariant representations in the form given some two years previously in Bargmann and Wigner [1948]. For the scalar wave equation, the correspondence with the representation $U_m^{+,0}$ acting on $\mathcal{H}_m^+ = L^2(\Theta^+, \mu)$ is immediate, as we have seen above.

Newton and Wigner proceeded as follows. They sought to define the position operators in terms of their generalized eigenstates, and to determine the latter, it was only necessary to construct *one* such generalized eigenstate (for convenience, localized at the origin). This function, denote f_0, then determines a function $f_{\mathbf{x}}$ localized at an arbitrary point $\mathbf{x}$ by application of the translation operator

$$(U_{1,\mathbf{x}} f_0)(p) = f_{\mathbf{x}}(p) \tag{8}$$

They then made the following postulates:

(A) All such functions f_0 form a linear space T_0.

(B) T_0 is stable under spatial rotations and the discrete symmetries.

(C) For any $f \in T_0$ and any $\mathbf{x} \neq 0$, $(U_{1,\mathbf{x}} f, f) = 0$.

(D) Certain regularity conditions.

Since obviously the elements of T_0 are generalized eigenfunctions and do not themselves lie in $\mathcal{H}_m^+$ there is a certain ambiguity in (C) (as also Eq.(8)); however from the explicit form of Eq.(8) given above it is obvious that no complications arise. (C) expresses the fundamental intuition, that a particle localized at the origin is orthogonal to a particle localized at any non-zero distance from the origin (all of these statements are to be

understood as applying to a fixed inertial frame of reference).

We consider the linear manifold of functions on Θ^+ stable under spatial rotations. For any integer j this manifold is spanned by the functions

$$P_m^j(\theta,\varphi)g(p_0,r) \quad m = -j,-j+1,\ldots,j-1,j \tag{9}$$

Where the P_m^j are spherical harmonics and (θ,φ,r) are spherical coordinates for $\mathbf{p} = (p_1,p_2,p_3)$; g is an arbitrary function. We wrote down explicit expressions for the discrete symmetries in (3.1.6)(Eq.7); the manifold Eq.(9) is already invariant under the parity transformation $\mathfrak{P} : f(p_0,p_1,p_2,p_3) \longrightarrow f(p_0,-p_1,-p_2,-p_3)$ and since by assumption it is invariant under time reversal

$$\mathfrak{T} : f(p_0,p_1,p_2,p_3) \longrightarrow \overline{f(p_0,-p_1,-p_2,-p_3)}$$

it follows that we may assume g is real. The inner product on $\mathcal{H}_m^+$ is given by Eq.(4) above and from (C) we conclude that

$$\int_{\mathbb{P}^3} \overline{f_0(p)}e^{-ip.x/\hbar}f_0(p)d^3p/p_0 = 0$$

and hence that $|f_0(p)|^2/p_0$ is constant. Comparison with Eq.(9) then shows that j=0, and since g may be chosen as real, we conclude

$$f_0(p) = p_0^{1/2}$$

up to a constant multiple. A generalized eigenstate of position localized at the point $(0,\mathbf{x})$ is then given by

$$f_{\mathbf{x}}(p) = e^{-ip.x/\hbar}p_0^{1/2} = e^{-ip.x/\hbar}(p^2+m^2c^2)^{1/4} \tag{10}$$

The regularity condition is only necessary to exclude the pathology, that the constant multiple may vary in sign for different values of p. We shall not discuss this here.

Newton and Wigner obtained analogous results for massive covariant wave equations of arbitrary (finite) spin; in the case of infinite spin, and zero-mass representations with s>1/2, no such notion of localization can be formulated satisfying (A),(B),(C),(D); the photon in particular is not localized in this sense. We shall not consider this further here. We study the Dirac theory in subsequent sections.

The position operator $\mathbf{x}_{op} = \mathbf{q}$ with components q^1, acting on

$f_{\mathbf{x}}$, must yield $x^i f_{\mathbf{x}}$. Clearly it is then the integral operator defined by

$$(q^i f)(p) =$$

$$(2\pi\hbar)^{-3}\int x^i e^{-i\mathbf{p}.\mathbf{x}/\hbar}(p_0\, p_0')^{1/2} e^{i\mathbf{p}'.\mathbf{x}/\hbar} f(p')d^3x d^3p'/p_0'$$

Since $\partial(p_0)^{1/2}/\partial p^i = p^i/2(p_0)^{3/2}$ it follows that

$$(q^i f)(p) =$$

$$\hbar(2\pi\hbar)^{-3}(i\partial/\partial p^i - ip^i/2p_0^2)\int (p_0)^{1/2} e^{i(\mathbf{p}-\mathbf{p}').\mathbf{x}/\hbar}(p_0')^{-1/2} f(p')d^3x d^3p'$$

or finally

$$(q^i f)(p) = i\hbar(\partial/\partial p^i - p^i/2p_0^2)f(p) \qquad (11)$$

It is a simple matter to verify that that this operator is self-adjoint in the inner product Eq.(4). To determine the corresponding operator acting on the functions φ on $\mathbb{R}^4$, we apply the transform Eq.(3) to find:

$$q^i\varphi(\mathbf{x},0) = x^i\varphi(\mathbf{x},0) + (8\pi)^{-1}\int \frac{e^{(-mc|\mathbf{x}-\mathbf{y}|/\hbar)}}{|\mathbf{x}-\mathbf{y}|}\,\partial\varphi(\mathbf{y},0)/\partial y^i d^3y \qquad (12)$$

The generalized eigenfunctions $f_{\mathbf{x}}$ go over to the functions

$$\varphi_{\mathbf{x}}(\mathbf{y},0) = 2^{-1/2}(2\pi\hbar)^{-3/2}\int_{\mathbb{P}^3} e^{i\mathbf{p}.(\mathbf{y}-\mathbf{x})/\hbar} p_0^{1/2} d^3p/p_0. \qquad (13)$$

Clearly q^i is a non-local operator, in terms of the dependence of φ on configuration space $\mathbb{R}^3$; and because of the factor $p_0^{-1/2}$ in the integrand of Eq.(13), $\varphi_{\mathbf{x}}(\mathbf{y},0)$ has non-zero configuration space extent; this function differs significantly from zero over a region of the order $\hbar/mc$; it is, however, non-zero over the *whole* of $\mathbb{R}^3$.

In view of this situation we introduce, following Segal [1964], [1967a], the terminology *q-local* for Newton-Wigner localized states. Locality with respect to configuration space will be called *c-local*. The precise distinction between these two notions of locality will become clear in what follows.

For future reference we note that Newton and Wigner determined the following expression for the position operator in the case s=1/2:

$$q^i = P^+(1+\gamma^0)\frac{p_0^{3/2}}{(p_0+mc^2)^{1/2}}\hbar(-i\frac{\partial}{\partial p_i})\frac{p_0^{-1/2}}{(p_0+mc^2)^{1/2}}P^+$$

where P^+ is the projection operator onto the positive

frequency solutions of the Dirac equation. This expression may be simplified to yield:

$$q^i = i\hbar\frac{\partial}{\partial p_i} + i\hbar\gamma^0\gamma^i/2p_0 - i\hbar\,\frac{\gamma^0\gamma^j p_j p_i + i\epsilon_{lmi}\sigma^l p_m|\mathbf{p}|}{2p_0(p_0+mc)\,|\mathbf{p}|} \qquad (14)$$

In configuration space this operator is once again non-local (hence c-non-local).

The question naturally arises as to the possibility of defining a transformation, possibly even a unitary transformation, from $\mathfrak{h}^+$ (that is functions on spacetime), restricted to a spacelike hypersurface (which is, we repeat, fixed throughout), to functions on $\mathbb{R}^3$ such that the position operator $\mathbf{q}$ acts locally. At the risk of some confusion, it seems useful to distinguish the two function spaces by introducing a new terminology for configuration space (that is, $\mathbb{R}^3$) *when we consider functions on this space defined through this transformation from functions on a spacelike hypersurface of spacetime.* In this circumstance we shall call $\mathbb{R}^3$ **position space**. The functions defined in this way we shall call *functions on position space.* As we have formulated it there is no mathematical difference between configuration space and position space; the terminology is for convenience only, and only the two function spaces (functions on configuration space and functions on position space) are mathematically distinct. In (3.2.9) below we formulate a more precise sense in which the two spaces may be distinguished. Position space defines that representation, in which q-local operators and states are manifestly local.

Clearly such a transformation, if it exists, must be non-local. To explore this possibility, we now consider the remarkable transformation discovered by Foldy and Wouthuysen, in the case of the *Dirac* theory.

3.2.5. The Foldy transform: the Dirac eqaution.

The Newton and Wigner paper marked an upsurge of interest in the physical interpretation of the relativistic wave equations, that was to continue well into the 60's. In this the representation of the Dirac theory discovered by Foldy and Wouthuysen [1950] played an important rôle.

The authors began with a critique of the standard technique for generating the non-relativisic limit of the Dirac equation. We recall that this proceeds from the observation that for an electron at rest two of the four components vanish, and that these components remain small so long as the energy is small in comparison to the rest mass. One then approximately solves for these components (in terms of the "large" components) and by substitution obtains a pair of equations for the large components; this pair of equations is formally identical to the Pauli spin equations, as first obtained by Darwin [1928]. Foldy and Wouthuysen noted the difficulty that the Hamiltonian thus obtained is no longer self-adjoint in the presence of external fields, and that to obtain the higher order approximations one must always return to the full 4-component theory. More serious, in the authors' view, is that one does not thereby obtain a correspondence between observables in the non-relativistic limit, and in the Dirac theory; most particularly, in connection with the velocity operator (*cf.* (1.4.5)), "one can well ask how the operators which purportedly represent the same physical variable in the two theories can have such completely different properties." (The authors here consider the operator $c\gamma^0\gamma^1$ as the velocity operator, with non-commuting components and with eigenvalues $\pm$ c.)[10]

[10] It is as well to make clear that, in the view taken here, there is no such thing as the non-relativistic limit. There are Galilean covariant theories and there are Lorentz covariant theories, and for a certain class of initial data it may be that the theories of each class agree in an approximate sense; but the use of limiting procedures of the form $c \longrightarrow \infty$ (as also $\hbar \longrightarrow 0$) are physically meaningless. Where they can be given a mathematical meaning it is only confirmed that the limit is highly singular, and that the mathematical structure underlying the formalism changes in

If Foldy and Wouthuysen motivated their analysis by appeal to the ambiguities of such limiting procedures, the actual strategy which they followed is better understood in the following terms: cedeing that it is only the momementum space states that resemble the wave functions of the non-relativistic theory (*cf.* (3.2.2)), one can at least try to make *this* correspondence precise. In particular, the authors considered that it should be possible to express the positive frequency momentum space states of the Dirac theory as *two*-component spinors, as in the non-relativistic theory. This idea was hardly original; just such a correspondence had been defined long before by Van der Waerden [1929], in his spinor formulation of the Dirac equation. Van der Waerden essentially showed that the Dirac equation could be reduced to a pair of equations coupling two 2-component spinors; only one of these need be specified, in terms of which the other can then be defined. Foldy and Wouthuysen now found an entirely different way of expressing the Dirac theory in terms of 2-component spinors.

In the representation

$$\gamma^0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \gamma^1 = \begin{bmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{bmatrix}, \tag{15}$$

where σ^1 are the usual Pauli matrices (Eq.(61), (1.4.4), the lower components of a plane-wave positive frequency Dirac bi-spinor vanish as $\mathbf{p} \longrightarrow 0$; the upper components vanish as $\mathbf{p} \longrightarrow 0$ for the negative frequency case. The Schrödinger defintion of an "odd" ("even") operator is one which does (does not) connect
positive and negative frequency plane-wave solutions . Is it then possible to exhibit an even operator in diagonal form (in particular the Hamiltonian)? If so, clearly we must pass to a new representation.

Consider now the linear transformation (the FW

an uncontrolled way. In our view, the theory of group contractions developed by Inönu and Wigner [1953] merely makes clear how the group structure changes in the limit, providing a kind of taxonomy for the comparison of the various spacetime groups involved.

transformation):

$$\varphi \longrightarrow e^{iS}\varphi = \varphi_F, \quad H \longrightarrow e^{iS}He^{-iS} = H_F \tag{16}$$

where $S = (-i/2mc)\gamma^1 p_1 \omega(|p|/mc)$ with ω arbitrary. It follows that $H_F = e^{iS\gamma^0}e^{-iS\gamma^0}H$. After some manipulation one finds:

$$H_F =$$

$$c\gamma^0\left|mc^2\left[\cos\lambda\omega(\lambda) + \lambda\sin\lambda\omega(\lambda)\right] + \frac{c\gamma^1 p_1}{\lambda}\left[\lambda\cos\lambda\omega(\lambda) - \sin\lambda\omega(\lambda)\right]\right|$$

(where we have written $\lambda = |p|/mc$). If one now sets $\omega(\lambda) = \lambda^{-1}\tan^{-1}\lambda$ we eliminate the last term (which contains the odd part of H_F) and obtain

$$H_F = \gamma^0 c(m^2c^2/p_0 + |p|^2/p_0) = c\gamma^0 p_0. \tag{17}$$

This result is to be expected; the Dirac equation was constructed as the linearized square root of the KG equation, and the FW transformation Eq.(16) but makes its equivalence with the square root equation explicit - appropriate to the spin 1/2 case, and taking due account of the ambiguity up to sign. For arbitrary momentum the positive frequency solutions are now exhibited as bi-spinors for which the lower two components vanish (and in the negative frequency case, in which the upper two components vanish). This is obvious from the form of Eq.(17); equally one can take the FW transform of the positive (negative) frequency solutions and verify this directly.

The transformation that we have just exhibited is a special case of a more general transformation which we shall need later. We state it as a theorem.

Theorem 3.2.1.

Given an operator A of the form $A = \rho_1 B + \rho_2 C$ on a Hilbert space $\mathcal{H} = L^2(X, \mathbb{C}^n, \mu)$ (for n finite) where B and C commute and have the action of the unit matrix on $\mathbb{C}^n$, and where the ρ which satisfy $\rho_i^2 = 1$ are anticommuting operators on $\mathbb{C}^n$, then there exists the linear transformation:

$$A \longrightarrow UAU^{-1} = \rho_2(B^2 + C^2)^{1/2}$$

where $U = e^{\rho_2\rho_1\omega/2}$ and $\tan\omega = BC^{-1}$. If ρ_1, ρ_2, and ω are self adjoint then U is unitary. (The proof is an easy reworking of the manipulations involved in defining the FW transform).

In the case of the FW transform $B = |pc|$, $C = mc^2$, and $\rho_1 = i\gamma^0\gamma^i p_i/|p|$, $\rho_2 = \gamma^0$. We shall call this transformation (of which the FW transform is a special case) the **Foldy transform.**

The "square root KG equation" is obtained from the Hamiltonian H_F of Eq.(17):

$$i\hbar\frac{\partial}{\partial t}\varphi_F = \gamma^0(\hbar^2\Delta - m^2c^2)^{1/2}\varphi_F \qquad (18)$$

That is, one obtains the pair of equations

$$i\hbar\frac{\partial}{\partial t}\varphi_F^+ = (-\hbar^2\Delta + m^2c^2)^{1/2}\varphi_F^+$$

$$i\hbar\frac{\partial}{\partial t}\varphi_F^- = -(-\hbar^2\Delta + m^2c^2)^{1/2}\varphi_F^-$$

where $\varphi_F^{+(-)}$ is a *two*-component spinor, the two upper (lower) components of the positive (negative) frequency FW transform solution of the Dirac equation. The Dirac equation is now in Hamiltonian form. As for the Born interpretation, it is apparent that (with the positive and negative frequency parts thus distinguished) multiplication by x^1 does *not* mix positive and negative frequency solutions; $\varphi_F^{\pm}(\mathbf{x},0)$ determines the initial data for a solution $\varphi^{\pm}(\mathbf{x},t)$ of the Dirac equation, and so too will $x^1\varphi^{\pm}(\mathbf{x},0)$. Multiplication by x^1 is thus a linear operator on the manifold of positive (or negative) frequency F transform solutions of the Dirac equation; however there still remains the question of whether this operator is self-adjoint on the two (separate) manifolds. From the inner product Eq.(6) we see that the F transform is unitary and that the new inner product on $\mathfrak{h}^+\cup\mathfrak{h}^-$ is just $(\varphi_F,\psi_F) = \int\Sigma\overline{\varphi_F(x)}\psi_F(x)d^3x$; multiplication by x^1 is clearly self-adjoint. Confirmation of this is immediate from the inverse F transform of multiplication by x^1; it is simple to verify that

$$e^{-iS}(i\hbar\frac{\partial}{\partial p_i})e^{iS} = i\hbar\frac{\partial}{\partial p_i} + i\hbar\gamma^0\gamma^i/2p_0 - i\hbar\,\frac{\gamma^0(\gamma^j p_j p_i + i\epsilon_{1mi}\sigma^l p_m|p|)}{2p_0(p_0+mc)|p|}$$

in agreement with the position operator q^i found by Newton and Wigner (see Eq.(14)). Consequently, the F transform of the spacetime solutions φ of the Dirac equation may be described as *position space* wave functions; for these the Born interpretation is directly applicable as in the non-relativistic theory.

3.2.6. The Foldy transform: KG equation.

Before considering the interpretation of these results we review the analogous theory of the KG equation. This theory was first developed by Case [1954] and elaborated by Foldy [1956] and Feshbach and Villars [1958][11]; the treatment below follows Case [1954].

We begin with the KG equation in the two-component form given by Kemmer [1939], Heitler [1943], and Sakata and Taketani [1940]; the idea is to reduce the KG equation to Hamiltonian form by incorporating the extra degree of freedom (present due to the second order time derivative) into a Hamiltonian equation determining the evolution of a *two*-component function (by analogy, the F transform applied to the Dirac equation leads to a two-component wave function, each component of which is a spinor). Defining then $\xi = i\hbar/mc^2 \frac{\partial}{\partial t} \varphi$ the KG equation may be written:

$$-i\frac{\partial}{\partial t}\varphi + mc^2/\hbar\ \xi = 0$$
$$\Delta\varphi + im/\hbar\ \frac{\partial}{\partial t}\xi - (mc/\hbar)^2\varphi = 0. \qquad (19)$$

Introducing the linear combination

$$\phi = 2^{-1/2}(\varphi+\xi)$$
$$\chi = 2^{-1/2}(\varphi-\xi)$$

allows us to write Eq.(19) in the more symmetric form:

$$i\hbar\frac{\partial}{\partial t}\phi = -(\hbar^2/2m)\Delta(\phi+\chi) + mc^2\phi$$
$$i\hbar\frac{\partial}{\partial t}\chi = (\hbar^2/2m)\Delta(\phi+\chi) - mc^2\chi$$

or defining the two-component function

$$\Psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}$$

and using the Pauli spin matrices, we obtain:

$$i\hbar\frac{\partial}{\partial t}\Psi = \left[(-\hbar^2/2m)(\sigma^3 + i\sigma^2)\Delta + mc^2\sigma^3\right]\Psi. \qquad (20)$$

[11] These authors did not acknowledge the work of Case [1954] or Foldy [1956], although both these papers appeared in the Physical Review. Feshbach and Villars claimed to eliminate the interpretative problem of negative energy states, whilst Foldy was primarily concerned to point out the possibility of using novel definitions of the inversion operators; it would be misleading to suggest there is uniformity in the interpretation of the theory.

This equation is to be considered the (non-covariant) scalar analogue of the Dirac equation. The Hamiltonian may be written in the form

$$H = \sigma^3(mc^2 + p^2/2m) + \sigma^2(ip^2/2m)$$

so that Theorem 3.2.1 can be immediately applied to yield:

$$H_F = \sigma^3\{(mc^2 + p^2/2m)^2 + (ip^2/2m)^2\}^{1/2}$$

$$= \sigma^3(p^2c^2 + m^2c^4)^{1/2} = \sigma^3 cp_0.$$

Here the linear transformation $U = e^{\rho_2\rho_1\omega/2}$ is given by $U = e^{\sigma^3\sigma^2\omega/2}$, where $\tan\omega = \frac{ip^2/2m}{(mc^2+p^2/2m}$; unlike the spin 1/2 case, ω is no longer self-adjoint but changes sign under Hermitian conjugation. As an operator on a Hilbert space of the form $L^2(X,\mathbb{C}^2,\mu)$ U cannot therefore be unitary. However, in the present discussion, it is by no means clear that we should consider the action of U on the Hilbert space $\mathfrak{h}^+ \cup \mathfrak{h}^-$; in particular, in passing from the KG equation to Eq.(20) the sesquilinear form of Eq.(2) goes over to

$$(\Phi,\Psi) = \int \bar{\Phi}\sigma^3\Psi \, d^3x \qquad (21)$$

and, in analogy to the Dirac case, one expects that the transformation above will preserve this covariant sesquilinear form, rather than the inner product that we obtain by reversing its sign for the negative-frequency solutions.

This difficulty does not arise for the Dirac equation, because the covariant sesquilinear form is positive definite and may be identified with an inner product on the entire manifold of solutions to the Dirac equation. That is not possible in the case of the KG equation. In the latter case, we find that indeed $(\Phi,\Psi) = (\Phi_F,\Psi_F)$ and that the Foldy transform preserves Eq.(21); this circumstance is usually expressed by the statement that U is *unitary with respect to the inner product* Eq.(21), and that $U = \sigma^3U^{-1}\sigma^3$ (these statements are meaningless; Eq.(21) is not an inner product, and neither unitary nor adjoint operators are defined on a space of indefinite norm)[12].

[12] Feshbach and Villars [1958] make great play of the fact that the expectation value of the Hamiltonian in the norm Eq.(21) is positive definite; but precisely the same is true for the usual invariant norm Eq.(2) (because whilst the

It is possible to eliminate the difficulty of the indefinite norm as follows: one simply takes the Hilbert space of positive and negative frequency solutions ($\mathfrak{h}^+$ and $\mathfrak{h}^-$), passes via the covariant Fourier transform to the momentum space $\mathcal{H}^+$ and $\mathcal{H}^-$, and defines the unitary transformation $\tau^{\pm}: \mathcal{H}^{\pm} \longrightarrow L^2(\mathbb{P}^3, dp^3)$ by:

$$(\tau^{\pm} f)(\mathbf{p}) = (\mathbf{p}^2 + m^2c^2)^{-1/4} f(\pm(\mathbf{p}^2 + m^2c^4)^{1/2}, p_1, p_2, p_3) \qquad (22)$$

Clearly τ maps the configuration space functions to the position space functions. However one must conclude from the foregoing that one *cannot* in this way unitarily transform the appropriate Hamiltonian equation (that is, $\pm c(\mathbf{p}^2+m^2c^2)^{1/2}\Phi_F^{\pm} = i\hbar\frac{\partial}{\partial t}\Phi_F$) into Eq.(20) (and thereby to the KG equation).

On the other hand, there are benefits which derive from just this feature of the formalism; namely, one can seek to generalize the Foldy transform from the free KG equation to the corresponding equation for an external field, because the invariant indefinite norm is derived from the covariant 4-current, unlike the norms on $\mathfrak{h}^{\pm}$, or $\mathfrak{h}^+ \cup \mathfrak{h}^-$.

Otherwise, the parallels with the FW representation are complete; in this new representation, the lower (upper) component of Φ_F once again vanishes for positive (negative) frequency components and the linear transformation defined by multiplication by $\mathbf{x}$ is once again the Foldy transform of the position operator defined by Newton and Wigner. We may therefore think of the Φ_F as functions on *position space* to which a Born interpretation is immediately applicable.

"energy" is negative, so too is the norm of the negative frequency solutions). As we noted in (1.4.4) (following Eq.(53)), it is just this fact which makes the total energy of the scalar field positive.

3.2.7. Interpretation.

The position space wave functions are defined, in a non-local way, from the spacetime wave functions. When one extends the Foldy transform (in a perturbation expansion) to the interacting case (see, in particular, Foldy and Wouthuysen [1950], Feshbach and Villars [1958]), one is confirmed in the suspicion that an external c-number field acts as a *non-local* perturbation on the position space wave functions. Of course one can introduce a perturbation, as a function *on position space*, at the level of the transformed equations in Hamiltonian form. Such a perturbation will act q-locally "at the position co-ordinates of the particle", and c-non-locally (via the inverse Foldy transform) at the level of the covariant equations. In this case the perturbation, regardless of its magnitude, can never induce transitions from positive to negative frequency solutions; in the former case, because multiplication by x induces such transitions, so too will a multiplicative operator which is a function of x.

Evidently one is confronted with a fundamental schism between the expression of the notion of local action in *position space* and the expression of local action in *configuration space.* It seems a q-local theory in position space has not been developed in non-trivial applications; no-one has produced arguments for or against such a theory. The metaphysical question, of which notion of locality is primary, has been answered in default in favour of the configuration space fields. One sees the primacy of group theory: it is only on spacetime that one has a simple group action of the *entire* inhomogeneous Lorentz group; and only the functions on configuration space are simply (and most paritucularly *locally*) related to the functions on spacetime (by restriction). Correspondingly, the functions on position space have no simple transformation properties under the entire group: the Foldy transform, like the Newton Wigner localized states, is on the face of it non-covariant; throughout we have fixed a definite inertial frame, which is left invariant under the Foldy transform, and with respect

to which the position space is defined. As it stands, *the concept of position space cannot be extended to provide a G-space for the Lorentz group* (but carries a group action *only* for the Euclidean sub-group, together with the inversions).

This fact was emphasized by Newton and Wigner, who remarked:

> a state which is localized at the origin in one coordinate system, is not localized in a moving coordinate system, even if the origins coincide at t=0. Hence our operators q have no simple covariant meaning under relativistic transformations. ([1949 p.403]).

I suggest the following interpretation for this circumstance; the position space function for a particle q-localized at the point x will, in configuration space, be c-localized over a region (about the point x) of spacial extent of the order of the Compton wavelength (and zero time-like extent). On an *active* interpretation of a Lorentz transformation, which leaves the point x invariant, one must simply conclude that imparting a definite velocity to a q-localized state destroys its q-local properties; there appears nothing counter-intuitive about this assertion, since although such a boost leaves x invariant (and hence the value of the actively transformed function φ' at the point x'=x), the q-localization properties of φ (and φ') depend not only on its values at the point x but also its values at space-like distances from the point x, and for these $\varphi' \neq \varphi$.

From a *passive* point of view, the situation appears slightly more counter-intuitive; for one describes one and the same physical situation in two co-ordinate systems (appropriate to two observers in relative motion, on the kinematic interpretation of symmetry transformations discussed in (2.5.2)). If this physical situation is that of: a particle q-localized at the point x, then one concludes both observers must describe this physical system so as to draw this same conclusion. This difficulty is resolved by the observation already made that in describing the q-local properties of such a system one must fix a space-like hypersurface relative to which q-locality is defined. So

long as the moving observer chooses the *same* space-like hypersurface, in terms of which the q-locality properties are defined, there will be agreement; it is only because the "natural" space-like hypersurface of the moving co-ordinate system is rotated relative to that of the initial co-ordinate system, that there is an apparent dependence on the frame of reference.

The decisive point in all three interpretations is therefore that the statement "a particle is localized at the point x" is ambiguous; one must also specify the inertial frame with respect to which the notion of q-locality is defined. For a comprehensive elaboration of this point we refer to Fleming [1965], [1966], [1981], who develops a formalism in which the inertial frame is explicitly indicated (one thereby obtains "hyperplane dependent localized wave functions").

The remarkable feature of relativistic quantum theory, that two entirely distinct state spaces define the physical interpretation of the kinematics, warrants a more fundamental perspective. We therefore consider the theory of localization from the point of view of the imprimitivity theory[13].

In (3.2.2) we pointed out that in order to obtain spacetime observables it is necessary to obtain a system of imprimitivity with spacetime space as the base of the associated projection-valued measure. Further, the concept of localization appropriate to a particle system, which we suppose to endure through time, cannot be formulated in terms of a projection-valued measure with $\mathbb{R}^4$ as base; we must choose $\mathbb{R}^3$ instead, and as a result we cannot obtain a representation of the full inhomogeneous Lorentz group. The largest subgroup which admits a representation of this form is then the Euclidean group.

[13] The arguments of Newton and Wigner, concise and accurate though they be, are not very perspicuous; it was Wightman who systematically elaborated the theory (Wightman [1962]) in the context of the general theory of imprimitivity.

The concept of localization is thus defined in terms of the existence of a system of imprimitivity for $\mathcal{E}$ based on $\mathbb{R}^3$ (actually we use the universal covering group $\mathcal{E}^*$ so as to subsume projective representations with exact multipliers under the imprimitivity theory). We can now give a more definite geometric meaning to the distinction between position space and configuration space, whilst at the same time making clear the consistency condition that must be satisfied, if a given representation of a spacetime group S^* is to admit this expression of locality (q-locality).

We consider $\mathbb{R}^3$ a G-space for $\mathcal{E}^*$; this space and its dual $\mathbb{P}^3$ defined in this way we call **position space** (and the position space dual). We also consider $\mathbb{R}^4$ a G-space for S^*; this space (and its dual $\mathbb{P}^4$) defined in this way we call **spacetime** (momentum space) and a spacelike hypersurface (and its dual) in this space **configuration space** (momentum 3-space). We suppose we have a (weakly continuous, unitary) representation V of $\mathcal{E}^*$ with associated projection valued measure Q based on position space; this means that $V_{h,\mathbf{a}} Q_E V^{-1}_{h,\mathbf{a}} = Q_{\delta(h).\,E+\mathbf{a}}$, which imposes certain constraints on this system of imprimitivity for $\mathcal{E}^*$, amongst all the possible systems of imprimitivity which arise from projection valued measures *based on the position space dual.* Now any representation of S^* on $\mathcal{H}$ will be associated with a system of imprimitivity for S^* based on momentum space; we denote this system U,P. By assumption $\mathcal{E}^*$ is a subgroup of S^*; therefore the restriction of U to $g \in \mathcal{E}^*$ will define a representation of $\mathcal{E}^*$; *we say the system* (U,P) *is* **localizable** *at the time* $x^0 = 0$ *if there exists a projection valued measure* T *based on* $\mathbb{R}^3$ *such that*

$$U_{h,a} T_E U^{-1}_{h,a} = T_{\delta(h).\,E+\mathbf{a}} \qquad (23)$$

for all $(h,a) = (h,(\mathbf{a},0)) \in \mathcal{E}^*$. Essentially, Eq.(23) requires that the restriction of the representation U of S^* to $\mathcal{E}^*$ is unitarily equivalent to one of the class of representations of $\mathcal{E}^*$ which satisfy the constraints mentioned above; that is, to be unitarily equivalent to a representation of $\mathcal{E}^*$ associated with a system of imprimitivity based on position space. When S is the inhomogeneous Lorentz group this consistency condition is

satisfied for the representations $U^{\pm,s}$ for m>0 defined in (3.1.6); it is *not* satisfied for the representation $U_0^{\pm,\pm s}$, $s \geq 1$, so that the photon is not localizable. For the Galilean group the unitary representations are not localizable (Inonu and Wigner [1952],[1953]); the projective representations with nonexact multipliers are localizable (Bargmann [1954], Wightman [1962]). Throughout the foregoing, this definition of locality is q-locality.

From the point of view of the distinction between position space and a hypersurface of spacetime, the fundamental point in the foregoing is that position space is defined with respect to the group $\mathcal{E}$, whilst a hypersurface of spacetime is defined with respect to the group S. Although $\mathcal{E}$ is a subgroup of S, it does not follow that the base space of the representation of $\mathcal{E}$ (position space) is pointwise isomorphic to the dual to the base space of the representation of $\mathcal{E}$ induced by the representation of S (momentum 3-space).

Incidentally, it follows (Wightman [1962]) from the general theory of homogeneous systems of imprimitivity that if a system is localizable, then the projection valued measure T can always be made to assume the canonical form

$$(T_E f)(x) = \chi_E(x) f(x) \qquad (24)$$

with $f \in L^2(\mathbb{R}^3, \mathcal{K}, d^3x)$ (which forms the Hilbert space of the restriction of U to $\mathcal{E}^*$), when U is a representation on the space $\mathcal{H} = L^2(\Theta, \mathcal{K}, \mu)$, with μ a quasi-invariant measure concentrated on the orbit Θ of S on momentum space. The canonical form of Eq.(24) may equivalently be characterized as that representation, in which the position operator acts multiplicatively. The Foldy representation is therefore in canonical form. The Hilbert space $L^2(\mathbb{R}^3, \mathcal{K}, d^3x)$ is the canonical Hilbert space of non-relativistic quantum mechanics. It might seem remarkable, that despite the fundamental differences between the non-relativistic and relativistic quantum theories, the two theories must have the same state space in the representation in which a Born (position) interpretation is immediately applicable. Like the CCR's, the non-relativistic state space is in some sense universal. We now see that this is a consequence of the fact

that both the spacetime groups contain the Euclidean group as a subgroup.

A final remark on interpretation. The Lorentz transformations were discovered through the study of the covariance properties of the free (classical) electromagnetic field, but the theory of Special Relativity was formulated from an epistemological standpoint. In the latter sense the theory in its original form was a theory about rigid rods and clocks, the possibilities of distant synchronization using "first signals". There can scarcely be greater ambiguity introduced into the foundations of the theory so conceived, than by the discovery that, insofar as the relativistic quantum theory admits any definition of the spacial location of a physical system, then the coordinate grid of events so defined is non-local relative to the Minkowski space on which the covariant mathematical description of interactions is formulated. From an epistemological point of view, space is position space, the space of q-localized events. From the point of view of the mathematical description, and central physical concepts about the nature of interactions and the co-ordinate independence of this description, physical space is configuration space.

It appears there is no way out of this dilemma; in relativistic quantum theory *both* notions of locality play a fundamental rôle. If what is truly local at the level of interactions is local in spacetime, and if relativistic quantum systems are all that there are in the world, then what we call local in the context of measurement theory (and thus *a fortiori* at the level of observable phenomena) is *in reality* non-local. Or one might say that the phenomenological world, existing on a fixed spacelike hypersurface, is non-locally determined by the microphysical world on that hypersurface. On a reductionist philosophy this conclusion appears inescapable; the alternative, that q-locality is prior, and that in reality the spacetime interactions are q-non-local, is then not available.

These conclusions follow from an elementary philosophy of measurement, which proceeds from the fundamental tenet that it is possible, relative to a given experimental context, to define the statistical distribution of a set of properties in correspondence with the observed macroscopic phenomena. In particular observed local properties must then correspond to the q-local properties of the microscopic quantum system. In Section 3.5 we propose a theory of measurement which applies the quantum theoretic description to all physical systems, whether "classical" (*i.e.* macroscopic), measurement apparati, or microscopic. It seems plausible that on this basis the fact that any system which could count as macroscopic will have an enormous (and effectively infinite) mass will reduce the non-local correspondence between position and spacetime locality[14]. Whether the non-locality can in this way be eliminated altogether is a problem, similar to the problem of "residual macroscopic coherence", which depends critically on the analysis of the use of idealizations (see (3.5.7)).

3.2.8. Relationship with the Wigner representations.

The Foldy-Wouthuysen transformation associates with each positive (negative) frequency solution of the Dirac equation a four-component bi-spinor in which the lower (upper) two components vanish, for *all* particle momenta. It is therefore trivial to effect a correspondence $\iota_{\pm}$ between positive or negative frequency 4-component solutions and 2-component positive or negative frequency spinors:

$\iota_{+}: (\alpha,\beta,\gamma,\delta) \longrightarrow (\alpha,\beta)$

$\iota_{-}: (\alpha,\beta,\gamma,\delta) \longrightarrow (\gamma,\delta)$

for $\alpha,\beta,\gamma,\delta \in \mathbb{C}$. One suspects that these 2-component spinors must be simply related to the 2- component spinors used in the Wigner spin 1/2 representations, because clearly we can

[14] One can also eliminate the distinction between position and configuration space by taking the limit $c \longrightarrow \infty$; this strategy is unsatisfactory, for reasons indicated earlier. Needless to say, the Compton wavelength is invariant whatever the velocity of a particle.

use the inverse of the unitary transformation $\tau_\pm$ defined by Eq.(22) to map the momentum space 2-component spinors, elements of the space $L^2(\mathbb{P}^3,\mathbb{C}^2,d^3p)$, isometrically onto the space $L^2(e^\pm,\mathbb{C}^2,d\mu^\pm)$ - the Hilbert space $\mathcal{H}_m^{\pm,1/2}$ of the Wigner representation $U_m^{\pm,1/2}$. Because further

$\tau_\pm \iota_\pm H_F \iota_\pm^{-1} \tau_\pm^{-1} = \pm(p^2c^2+m^2c^4)^{1/2} = \pm H$

where H_F is the F transform of the Dirac Hamiltonian Eq.(16):

$H_F = \gamma^0(p^2c^2+m^2c^4)^{1/2}$.

$\pm$ H therefore generates the unitary time-evolution

$f(p) \longrightarrow e^{\mp icp_0 t/\hbar} f(p)$

on elements of $\mathcal{H}_m^{\pm,1/2}$, in agreement with the evolution defined by $U_m^{\pm,1/2}$ (cf. (3.1.4) fn. 1).

To fully establish this result (due to Foldy [1956], Fonda and Ghirrardi [1968], [1970]), we need to show that if f_W, f_F, f_D are positive frequency momentum space states in the Wigner, Foldy-Wouthuysen, and Dirac representations respectively, then the sequence of transformations:

$f_W - \iota_+^{-1}\tau_+ \longrightarrow f_F - F^{-1} \longrightarrow f_D$

transforms the strict cocyle $\phi^{1/2}(m,\delta(m)^{-1}.p)$, $m \in SL(2,\mathbb{C})$, $\phi^{1/2} = \mathcal{D}^{1/2}$ (the 2-dimensional representation of SU(2)) into the 4-dimensional non-unitary matrix S(m) under which the Dirac bi-spinor f_D transforms. The same is required (with ι_-,τ_- in place of ι_+,τ_+) for the negative frequency states.

We shall not work through the explicit calculations; we refer to Fonda and Ghirrardi [1970 5.7.2] for a representation-dependent calculation. The upshot of this discussion is that for spin 1/2 particles we have *three* distinct momentum-space representations; the Wigner, Foldy-Wouthuysen, and Dirac representations. The Foldy-Wouthuysen representation, a 4-component representation of the form $\begin{bmatrix} \tau_+ f_W^+ \\ \tau_- f_W^- \end{bmatrix}$, where $f_W^\pm \in \mathcal{H}_m^{\pm,1/2}$, is simply related to the position space representation (it is the direct sum of the positive and negative mass hyperboloid position space representations) and we may instead consider the fundamental representations as the canonical, position space, and covariant respectively; all three exist for

abitrary spin. In position space the position operator always takes the form $(x^i_{op}f)(\mathbf{p}) = i\hbar\frac{\partial}{\partial p_i}f(\mathbf{p})$ (and is multiplication by x^i in the Fourier transform space); because the canonical and position space representations are related (for arbitrary spin) by the isomorphism $\tau^{\pm}$ the position operator always takes the form $\tau_{\pm}(i\hbar\frac{\partial}{\partial p})\tau_{\pm}^{-1}$ in the canonical representation, which is the familiar position operator for the scalar case Eq.(11).

A final remark; the relationship between the canonical (or position space) representations and the covariant representations remains obscure; the FW transformation is not perspicuous. The one insight which it appear to offer, that the Dirac representation employs 4-component spinors because one takes the direct sum of representations on the positive and negative mass hyperboloids before applying the inverse FW transformation

$$\begin{pmatrix} \tau_+ f^+_W \\ \tau_- f^-_W \end{pmatrix} \xrightarrow{\quad F^{-1} \quad} f_D$$

is actually misleading. Of course we know that the Dirac equation admits positive and negative frequency solutions, so that any algorithm which associates cannonical states with an arbitrary solution of this equation will in general define a pair of canonical states, one from each mass hyperboloid. The FW transformation emphasises this feature of the correspondence at the expense of others. In particular we do not learn why a positive frequency covariant state must be represented as a 4-component object; as we shall see, it is not a consequence of covariance under the ILG (and still less of combining positive and negative frequency representations).

3.3. Canonical Second Quantization

.....but second quantization is a functor.
E. Nelson

3.3.1. Introduction.

Clearly the fundamental property of the relativistic wave equations is covariance in spacetime [1]. Because of its importance it would be useful, and perhaps even essential, to define this concept abstractly. We cannot do this is a fully satisfactory way; for the local systems of (2.5.3) covariance means that the local algebras[2] (with a limiting finite basis of fields, in the sense of Fredenhagan and Hertel [1981], for example), have a structure which is independent of space time coordinates, and which is described in an intrinsic way independent of any spacetime orientation or spatial rotation. Therefore an actively interpreted spacetime transformation can do little more than modify the way each (essentially identical) local algebra is associated with a choice of inertial frame.

The "intrinsic description" is essentially a choice of an equivalence class of functions (or quantum fields) which are connected by a linear transformation, under which they form a closed linear manifold. In conventional QFT the manifold is finite dimensional and the transformations are non-unitary. The Dirac field, (or equally the Dirac wave

[1] As we have seen there can be no covariant representation on position space $\times T$.

[2] By this we mean c-local algebras, the covariant local algebras. We shall consider q-local algebras in (3.4.8).

functions), the most important example, transform as $\psi \longrightarrow S(m)\psi$ where $S(m)$ is a non-unitary 4-dimensional matrix representation of $SL(2\mathbb{C})$.

From the point of view of rotations, this idea corresponds exactly to the intuition that the numerical values of the wave function (for simplicity) at a point in spacetime define a tensor, with repect to a coordinate system. These functions define geometric objects at each point in spacetime. This intuition is not directly applicable to NRQM wave functions on configuration space; the massive representations are all projective and the complex phase is one of the things that is rotated under a boost.

There is a general way of relating a unitary representation of a spacetime group G on a complex Hilbert space to a representation in which the action is broken into two parts; the one, acting on an internal space for fixed $(\mathbf{x},t)$, the other mapping these spaces into each other. Correspondingly the wave function or quantum field is conveniently represented by an n-component function on $\mathbb{R}^4$, spanning an n-dimensional spin space. The solution manifold of the Dirac equation defines an irreducible representation of this type where the spin space is very special indeed; it is the 4-dimensional representation space for the Clifford algebra (unique up to unitary equivalence), and it carries an essentially unique representation of $SL(2,\mathbb{C})$. There are other ways of relating the $m> 0$, $j=1/2$ Wigner representations to covariant wave equations. This particular choice is the unique covariant irreducible one, which also extends to the inversions.

In (3.3.6) the general construction in Mackey theory for imbedding a Wigner representation in a spin representation is discussed. The appropriate state space is a Hilbert space bundle, with spin space as the fibre over spacetime or the one or two sheeted mass-hyperboloid as base. This embedding is actually implicit in the standard formalism of RQFT, particularly the plane wave expansion of the free fields, and the momentum space Dirac spinors which accompany the

creation and annihilation operators are really transition matrices connecting the two types of representation. This will become evident. In the 1-particle theory, one has something formally similar, namely the plane wave expansion
$\psi(x) = \sqrt{2}(2\pi)^{-3/2}\Sigma_\sigma \int e^{-ip.x/\hbar} u(p,\sigma)b(p,\sigma)\, d\mu^+$
but here the objects $u(p,\sigma)b(p,\sigma)$ span the state space, the $\mathbb{C}$-valued functions $b(p,\sigma)$ are not associated with another Hilbert space carrying a different representation. The standard formalism, in which the b's are taken as annihilation operators acting on a Fock space $\mathcal{H}$ on which a unitary representation of the ILG acts, in fact requires that this representation is the canonical second quantization of the Wigner j=1/2 representation ("canonical" in the sense of (1.3.4)). The bi-spinors with components $u(p,\sigma)^1$ may be defined as those 4×2 matrices $u(p)^{1\sigma}$ which transform the unitary 2-dimensional $D^{1/2}$ matrix representation of the stability sub-group SU(2) which occurs in the Wigner transformation Eq.(2), (3.1.5) to the non-unitary 4-dimensional matrix representation S of $SL(2,\mathbb{C})$. At the same time, as 4-component objects, the $u(p,\sigma)$ (for fixed p) span a 2-dimensional sub-manifold of the 4-dimensional fibre $\mathbb{C}^4$ sitting over the point p of Θ^+; the "negative energy" bi-spinors $v(p,\sigma)$ span a similar sub-manifold over the point p of Θ^-. Obviously the u's and v's must be constrained for this to be so, and the constraints on each are provided by the Dirac equation. These sub-manifolds are irreducible representations for the stability group SU(2) at ±p, on which the representation S and $D^{1/2}$ coincide. The inversions do *not* interchange these sub-manifolds. The distinction between representations on Θ^+ and Θ^- is preserved in the covariant representation thus defined.

The representation on Θ^- is *not* employed in the sequel. The existence of antiparticles, as a distinct but group theoretically identical representation to the particles, is forced by other considerations, namely *microcausality* and *gauge-covariance* of the spacetime fields.

To make this circle of ideas really intelligible there seems

no option but to sketch, at least schematically, the basic theory. In particular to understand the group theoretic interpretation of the plane-wave expansion, which is a basis dependent formula, it is necessary to work with generalized eigenfunctions and the point fields associated with this (overcomplete) set of "basis vectors" with some loss of rigour as a consequence. This formalism is presented first; the construction is due to Weinberg [1964], slightly modified in the manner of Novozhilov [1975 5.1]. In (3.3.6) we then rework some aspects of this construction in a more abstract setting. The remainder of this section is concerned with interpretation.

Our primary goal is to understand the plane wave expansion of the standard formalism. As detailed in (1.4.7), we wish to understand: why the particle interpretation provided by this expansion must refer to two types of particles, oppositely charged but otherwise identical; the relationship to negative energy states; and the relationship to the 1-particle theory. The standard formalism, in particular the plane wave expansion of the spacetime fields, plays an absolutely fundamental rôle in all heuristic interpretations of RQFT (or at least those theories which admit a scattering situation). One wants to know whether and in what sense this plane wave expansion is unique, to know what assumptions make it unique.

In this context, knowing the precise details of the relationship between the Wigner representations and the covariant representations is fundamental, because it permits a step by step evaluation of the assumptions that must be made, from the canonical basis of NRQM, to fabricate (and thereby interpret) such an unnatural object as the covariant quantum field (complete with plane wave expansion). Essentially this defines a particle interpretation of the fields, so that these assumptions are implicit even if one starts off from a Lagrangian field theory. The Lagrangian theory gives the commutation relationships between the fields, and defines the generators of the spacetime transformations (and internal symmetry transformations) in

terms of the fields. It also suggests expressions for covariant interactions and non-linear generalizations. What it does not do is provide a particle interpretation, which is rather imposed, by adoption of the plane wave-expansion. As a result, and for the reasons given in (1.4.4), most of the obvious questions concerning negative energy states and their re-interpretation as positive energy antiparticle states remain unanswered. However the Lagrangian formalism naturally motivates the assumptions of covariance, microcausality and gauge covariance, so in this sense provides the most satisfactory starting point for the theory. Because these assumptions are not required in the canonical second quantization of NRQM, and because they have a tenuous interpretation from the point of view of the fundamental principles of NRQM, the Lagrangian framework is undoubtedly superior, quite apart from its utility in defining (if only figurative) interacting theories.

But even from the standpoint of Lagrangian theory the construction is elaborate; it will emerge that there is no direct formal connection between the negative energy states and the antiparticle states and that the relationship with the 1-particle theory on the two-sheeted mass hyperbola remains obscure. For these reasons we develop an altogether different quantization in Section 3.4. All of the difficulties which we shall encounter below are eliminated in this quantization, which applies with minor differences to the relativistic and non-relativistic field theories alike.

To conclude this introduction we formulate the difficulty of defining spacetime fields with local transformation properties.

Recall that the representations $U^{+,j}_m$ acting on $\mathcal{H}^{+,j}_m = L^2(\Theta^+, \mathcal{K}^j, \mu)$ with μ^+ a quasi-invariant measure which lives on Θ^+ have the form (Eq. (5) (3.1.5)):

$$(U_{h,x}f)(p) = r_{\delta(h).p}(\delta(h)^{-1}.p)e^{ip.x/\hbar}\phi^j(h,\delta(h)^{-1}.p)f(\delta(h)^{-1}.p) \quad (1)$$

where for $h \in K^* \simeq SU(2)$ (the stability sub-group at the point $\overset{\circ}{p} = (mc, 0, 0, 0)$), $\phi^j(h, \delta(h)^{-1}\overset{\circ}{p}) = \phi^j(h, \overset{\circ}{p}) = D^j(h)$, where D^j

is the 2s+1 dimensional irreducible representation of the rotation group SU(2).

Suppose now that we choose the invariant measure μ^+ so that we can eliminate the Radon-Nikodyn derivative r_g in Eq.(1); apart from the translation term $e^{ip.x/\hbar}$ we see that the strict $(e^+, \mathcal{K}, \cdot)$ cocycle ϕ^j depends on the momentum p; if then we were to attempt to define the creation operator $a(p)^*$ on $\mathfrak{F}(\mathcal{H})$ with $\mathcal{H}$ as above, the Hilbert space of states transforming under Eq.(1), it follows that its transformation law will include a matrix $A(h,p)$ in M which depends on p (we shall shortly construct this matrix from the cocycle ϕ^j):

$$U_{h,x}a(p)^*U_{h,x}^{-1} = e^{i(\delta(h).p).x/\hbar}A(h,p)a^*(\delta(h).p). \qquad (2)$$

Defining the spacetime creation field Φ^* (the sign of the exponent is conventional[3]):

$$\Phi^*(x) = (2\pi\hbar)^{-3/2}\int_{\mathbb{P}^3} e^{ip.x/\hbar}a^*(p)\, d^3p/\sqrt{2p_0} \qquad (3)$$

will lead to the transformation:

$$U_{h,a}\Phi^*(x)U_{h,a}^{-1} = \qquad (4)$$
$$(2\pi\hbar)^{-3/2}\int_{\mathbb{P}^3} e^{i(\delta(h).p).(\delta(h).x+a)}A(h,p)a^*(\delta(h).p)d^3p/\sqrt{2p_0}$$

Because of the p dependence of the matrix A there seems no hope of obtaining a transformation law of the form

$$(2\pi\hbar)^{-3/2}\int_{\mathbb{P}^3} e^{ip.(\delta(h).x+a)}S(h)a^*(p)\,d^3p/\sqrt{2p_0} \qquad (5)$$

(with S(h) a matrix representation of the Lorentz group on $\mathcal{K}^j$), from which one concludes:

$$U_{h,a}\Phi^*(x)U_{h,a}^{-1} = S(h)\Phi^*((\delta(h),a)^{-1}.x) \qquad (6)$$

characteristic of the local spacetime fields.

The question of the relationship of the Wigner representations to quantum fields was first posed in this form by Weinberg [1964]; the relationship with the covariant

[3] I do not wish to pretend that this question is altogether trivial; this convention is related to sign conventions for the Schrodinger equation and the Wigner transformations, and it determines the corresponding sign in the exponent for the annihilation operator. A thorough analysis of sign conventions in RQT is beyond the scope of this thesis, so I say as little as possible about it. A consistent sign convention is used throughout, however.

wave equations was first established by Joos [1962]; the general theory was implicit in Mackey's [1953] generalization of the Frobenius reciprocity theorem and developed in an explicit physical context by Hermann [1966]. For a complete and self-contained treatment we refer to Varadarajan [1970]; treatments along the lines of Joos [1962] are readily available (e.g. Halpern [1968], Novozhilov [1975], Barut and Raczka [1977]), whilst the general theory of group representations on fibre bundles has become a vast field in pure mathematics.

In the sequel we use natural units $\hbar = c = 1$.

3.3.2. The scalar field.

The scalar case j=0, for which the cocycle ϕ^j is the identity, is obviously exceptional. In this case Eq.(4) and Eq.(6) are compatible, with S the identity representation. We may now canonically second quantize this representation, defining $a^*(f)$ in the canonical sense of (1.3.4) for $f \in \mathcal{H}_m^{+,0}$ and $a^*(k)$ as a bilinear form which creates the generalized eigenstate of momentum k. Let us denote henceforward by f_k such generalized eigenstates (we use the symbol k to denote the generalized eigenvalue, and p to denote the argument of f_k as a function on Θ^+). If we now want to write the expansion formula for arbitrary $g \in \mathcal{H}_m^{+,0}$:

$$g = \Sigma_i f_{k_i}(f_{k_i}, g) \tag{7}$$

in a manifestly covariant way:

$$g = \int_{\Theta^+} f_k(f_k, g)\, d\mu^+(k) \tag{8}$$

then it is necessary to normalize the f_k's to:

$$(f_k, f_{k'}) = k_0 \delta^3(\mathbf{k}-\mathbf{k}') \tag{9}$$

(where $k_0 = (\mathbf{k}^2 + m^2)^{1/2}$). Therefore f_k can be written:

$$f_k(p) = k_0 \delta^3(\mathbf{k}-\mathbf{p}). \tag{10}$$

f_k is a distribution of the form δ_m in the notation of (1.3.4) satisfying $\int \overline{f_k(p)} g(p) d\mu^+(p) = g(k)$; $\mathcal{H}_m^{+,0}$ is further an L^2 space so we may take over without modification the canonical second quantization and define the point creation and annihilation fields $a^*(f_k)$, $a(f_k)$ as in Eq.(48) (1.3.4); we denote these fields $a^*(k)$, $a(k)$ and from Eq.(49) (1.3.4) we see that:

$$[a(k),a^*(k')] = k_0\delta^3(\mathbf{k}-\mathbf{k}'). \qquad (11)$$

The a(k)'s, $a^*(k)$'s can be defined in a rigorous sense as quadratic forms on an appropriate domain D⊗D of $\mathfrak{F}(\mathcal{H})\otimes\mathfrak{F}(\mathcal{H})$ via Eq.(10); we shall not pursue this here.

The spacetime quantum fields are then defined as:

$$\Phi(x) = 2^{-1/2}(2\pi)^{-3/2}\int_{\Theta^+}a(k)e^{-ik.x}d\mu^+$$
$$\Phi^*(x) = 2^{-1/2}(2\pi)^{-3/2}\int_{\Theta^+}a^*(k)e^{ik.x}d\mu^+ \qquad (12)$$

These fields transform locally, by transport of the transformation properties of the creation and annihilation operators. However their commutator is

$$[\Phi^*(x),\Phi(x')] = 2^{-1}(2\pi)^{-3}\int_{\Theta^+}e^{-ik.(x-x')}d\mu^+ = \Delta^+(x-x')$$

which does *not* vanish when x-x′ is spacelike.

Note that we may instead second quantize the unitarily equivalent Hilbert space defined by the isomorphism $\tau^{\pm}$ of (3.2.6), which is the space $L^2(\mathbb{P}^3,d^3p)$, whereupon the change in the measure d^3p on $\mathbb{P}^3$ is compensated by the Radon-Nikodyn derivative which enters into the Wigner transformation. The formulae above must then be modified to take account of this; essentially the measure μ^+ is everywhere replaced by the invariant measure on $\mathbb{P}^3$ and the k_0 term in the normalization of the states and the CCR between the creation and annihilation operators is absent. This corresponds (for arbitrary spin) to a canonical second quantization in the position space dual. We shall consider this later.

Second quantizing the positive mass hyperboloid gives us a pair of covariant spacetime fields which create and destroy positive energy 1-particle states, but which do not satisfy microcausality. From the point of view of the canonical second quantization the transition to the physical scalar field, either real or complex, is unmotivated. The real physical scalar field is the linear combination of the creation and annihilation fields as defined above. It then satisfies microcausality. The complex physical scalar field is a combination of creation and annihilation fields acting on two disinct but (from a group theoretical point of view) identical Fock spaces (the particle and anti-particle Fock

spaces). The correct linear combinations also satisfy microcausality, and there is an additional gauge covariance present under which the fields transform as
$\Psi(x) \longrightarrow e^{i\theta}\Psi(x)$, $\Psi^*(x) \longrightarrow e^{-i\theta}\Psi^*(x)$
(where Ψ^* is the adjoint of Ψ).

Quite generally, we shall consider the antiparticle as an irreducible representation of the ILG on the *positive* mass hyperbola. It is group-theoretically identical with the particle. In order to continue we consider the general case[4].

3.3.3. The covariant quantum field; arbitrary spin.

The sequel can be applied to fields of arbitrary (finite) spin, but we shall explicitly have in mind the case s=1/2. We once more choose the invariant measure μ^+ on Θ^+, where $d\mu^+ = 2\delta(p^2-m^2)\theta(p_0) = d^3p/p_0$ on $\mathbb{P}^3$, and the representation

$$(U_{h,a}f)(p) = e^{ia.p}\phi^j(h,\delta(h)^{-1}.p)f(\delta(h)^{-1}.p) \qquad (13)$$

acting on the Hilbert space $\mathcal{H} = L^2(\Theta^+,\mathcal{K}^j,\mu^+)$ where $\mathcal{K}^j$ is a 2j+1 dimensional complex Hilbert space. ϕ^j is, we recall, a strict (G,Θ^+,M) cocycle with values in M, the unitary group of $\mathcal{K}^j$ (therefore for j=1/2 a unitary 2×2 matrix).

We fix the generalized eigenstate $f_0(p)^\sigma$ as the spinor describing an electron at rest with z-component of spin σ, $\sigma = \pm 1$. Explicitly,

$$f_0(p)^+ = \begin{bmatrix} 1 \\ 0 \end{bmatrix} m\delta^3(\mathbf{p}) \ , \ f_0(p)^- = \begin{bmatrix} 0 \\ 1 \end{bmatrix} m\delta^3(\mathbf{p})$$

We introduce the following notation; that hermitian element of SL(2,ℂ) which induces the boost $\overset{\circ}{p} = (m,0,0,0) \longrightarrow (p_0,\mathbf{p}) = p \in \Theta^+$ we call b(p). The generalized momentum eigenstates are to be generated from f_0 by the application of boosts via (13), that is,

$$f_k(p)=(U_{b(k),0}f_0)(p) = \phi^j(b(k),\delta(b(k))^{-1}.p)f_0(b(k)^{-1}.p) \qquad (14)$$

Using the unitarity of ϕ^j, one may verify directly the

[4] These statements will be justified in what follows. In the case of mass zero fields the particle and anti-particle Fock spaces may be interchanged by the parity inversion. We confine out attention to the massive case.

expansion formula (*cf.* Eq. (8)):

$$g(p') = \int \Sigma_\sigma f_k(p')^\sigma \overline{f_k(p)}^\sigma g(p)\, d^3p/p_0\, d^3k/k_0 \qquad (15)$$

(for $g \in \mathcal{H}$) as also the orthogonality relationships

$$(f_k^\sigma, f_{k'}^{\sigma'}) = k_0 \delta^3(\mathbf{k}\,\mathbf{k}')\delta_{\sigma\sigma'} .$$

Using the group action one may determine the action of $U_{h,0} = U_h$ on the basis states:

$$U_h f_k^\sigma = U_h U_{b(k)} f_0^\sigma = U_{b(h.k)} U_{b(h.k)}^{-1} U_h U_{b(k)} f_0^\sigma$$

$$= \int U_{b(h.k)} \Sigma_{\sigma'} f_{k'}^{\sigma'} (f_{k'}^{\sigma'}, U_{b(h.k)}^{-1} h b(k) f_0^\sigma)\, d^3k'/k'_0 .$$

Note that $b(h.k)^{-1} h b(k)$ is an element of the stability sub-group $G_0 \simeq SU(2)$ at the point $\mathring{p}$; we temporarily denote this element $\{h,k\}$ and using the orthogonality relation-ships one finds:

$$(f_{k'}^{\sigma'}, U_{\{h,k\}} f_0^\sigma) = k'_0 \delta^3(\mathbf{k}') \phi^j(\{h,k\}, \delta(\{h,k\})^{-1}.p)^{\sigma'\sigma}$$

so that finally

$$U_h f_k^\sigma = f^{\sigma'}_{\delta(h).k}\, \phi^j(\{h,k\}, \delta(\{h,k\}^{-1}).\mathring{p})^{\sigma'\sigma}. \qquad (16)$$

Since now for $h \in G_0$, $\phi^j(h, \delta(h)^{-1}\mathring{p}) = \phi^j(h,\mathring{p}) = D^j(h)$, where D^j is the $2j+1$ dimensional irreducible representation of the rotation group SU(2), we may replace the abstractly defined strict cocycle ϕ^j by this matrix representation and thus obtain

$$U_h f_k^\sigma = f^{\sigma'}_{\delta(h).k}\, D^j(\{h,k\})^{\sigma'\sigma} \qquad (17)$$

(note that for $j=1/2$, $D^{1/2}(\{h,k\}) = \{h,k\} \in G_0 \simeq SU(2)$).

We now define creation and annihilation operators $a^*(k)^\sigma = a^*(f_k^\sigma)$, $a(k)^\sigma = a(f_k^\sigma)$ in the canonical way, except that we do not specify whether these are to act on the symmatrized or antisymmetrized Fock space; they obey either the CCR or ACR, and we shall rarely need to explicitly indicate which of the two brackets are involved (that is, we shall let $[.,.]$ stand for either). No confusion will result. The transformation properties of the creation operator follows directly from Eq. (17):

$$U_{h,x} a^*(k)^\sigma U_{h,x}^{-1} = e^{i(\delta(h).k).x} D^j(\{h,k\})^{\sigma'\sigma}\, a^*(\delta(h).k)^{\sigma'} \qquad (18)$$

The annihilation operator must then transform as:

$$U_{h,x} a(k)^\sigma U_{h,x}^{-1} = D^j(\{h,k\}^{-1})^{\sigma\sigma'}\, a(\delta(h).k)^{\sigma'} e^{-i(\delta(h).k).x} \qquad (19)$$

(taking the adjoint and using the unitarity of D^j); of course we cannot define the quantum field simply as

$$\Phi^*(x)^\sigma = (2\pi)^{-3/2} \int e^{ik.x} a^*(k)^\sigma\, d^3k/\sqrt{2k_0}$$

because of the k-dependence of the D^j. However, following Weinberg [1964] we notice that if we extend the matrix

representation D^j of SU(2) to a non-unitary matrix representation $\mathcal{D}^j$ of $SL(2,\mathbb{C})$ such that $\mathcal{D}^j(g) = D^j(g)$ for $g \in$ SU(2), we may then write

$$D^j(\{h,k\}^{-1}) = D^j(b(k)^{-1}h^{-1}b(h.k)) = \mathcal{D}^j(b(k)^{-1})\mathcal{D}^j(h^{-1})\mathcal{D}^j(b(h.k)) \quad (20)$$

so that if we now define

$$a^-(k)^\sigma = \mathcal{D}^j(b(k))^{\sigma\sigma'} a(k)^{\sigma'} \quad (21)$$

one finds that

$$U_h a^-(k)^\sigma U_h^{-1} = \mathcal{D}^j(h^{-1})^{\sigma\sigma'} a^-(\delta(h).k)^{\sigma'}. \quad (22)$$

The transformation matrix now depends on h only. To achieve the *identical* transformation properties of a (suitably defined) creation operator (under *homogeneneous* transformations), it is necessary to put Eq.(18) into a similar form as Eq.(19). To this end we note that the complex conjugation of the representation D^j is unitarily equivalent to D^j; that is, for each j, there exits a unitary matrix C (Rose [1957 p.48]) such that:

$$\overline{D^j(g)} = C\, D^j(g)\, C^{-1}, \quad \bar{C}C = (-1)^{2j}$$

(when j = 1/2 we may take C as the matrix defined by Eq.(62) (1.4.4)). It follows that $D^j(g) = (CD(g^{-1})C)^t$ so that we may rewrite Eq.(18) as:

$$U_h\, a^*(k)^\sigma U_h^{-1} = (CD^j(\{h,k\}^{-1})C^{-1})^{\sigma\sigma'} a(\delta(h).k)^{\sigma'}. \quad (23)$$

We now define

$$a^+(k)^\sigma = (\mathcal{D}^j(b(k))C^{-1})^{\sigma\sigma'} a^*(k)^{\sigma'}$$

and from Eq.(20), (23) one obtains:

$$U_h a^+(k)^\sigma U_h^{-1} = \mathcal{D}^j(h^{-1})^{\sigma\sigma'} a^+(\delta(h).k)^{\sigma'}. \quad (24)$$

Covariant spacetime fields may now be defined as:

$$\Phi^-(x)^\sigma = (2\pi)^{-3/2}\int e^{-ik.x}(\mathcal{D}^j(b(k)))^{\sigma\sigma'} a(k)^{\sigma'} d^3k/\sqrt{2k_0} \quad (25)$$

$$\Phi^+(x)^\sigma = (2\pi)^{-3/2}\int e^{ik.x}(\mathcal{D}^j(b(k))C^{-1})^{\sigma\sigma'} a^*(k)^{\sigma'} d^3k/\sqrt{2k_0}.$$

Both fields transform locally as

$$U_{h,a}\Phi^\pm(x)U_{h,a}^{-1} = \mathcal{D}^j(h^{-1})\Phi^\pm(\delta(h).\, x+a\,)$$

but their (anti)commutator is non-local.

It will have been noticed that that Φ^+ is *not* the adjoint of Φ^- (nor a^+ the adjoint of a^-); the fields $\Phi^\pm$ were defined so as have the same transformation properties. The adjoint field to Φ^- is rather given by:

$$\Phi^{-*}(x)^\sigma = (2\pi)^{-3/2}\int e^{ik.x}(\overline{\mathcal{D}^j(b(k))})^{\sigma\sigma'} a^*(k)^{\sigma'} d^3k/\sqrt{2k_0}$$

and transforms as:

$$U_{h,a}\Phi^-(x)^* U_{h,a}^{-1} = \overline{\mathcal{D}^j(h^{-1})}\Phi^-(\delta(h).x+a)^*$$

that is, corresponding to the complex conjugate representation to Φ^-. In factoring the unitary infinite dimensional represention of SL(2,$\mathbb{C}$) to a covariant (still infinite dimensional) representation, in which a finite dimensional non-unitary representation acts on the spin space $\mathcal{K}$, one has a number of inequivalent choices; every *irreducible* representation is of the form $\mathcal{D}^j \otimes \overline{\mathcal{D}^{j'}}$ for j, j' = 0, 1/2, 1, ...(Weyl [1946]) (which we call the (j, j') representation). We have therefore made an arbitrary choice of the representation (j,0) for the annihilation field Φ^- (with the consequence that the adjoint field transforms according to the representation (0,j)). Naturally we can equally well define the fields $\Psi^\pm$ transforming according to the (0,j) representation, with adjoint Ψ^{-*} transforming according to the (j,0) representation[5].

This embarrasment of choice does not exist in the Galilean theory; there is *no* non-zero mass finite dimensional representation of this group.

3.3.4. Anti-particles, gauge invariance and microcausality.

It will be convenient in what follows to consider, instead of the (0,j) fields, rather the unitarily equivalent representation (0,j)~ defined by $\mathcal{D}^j(g^{-1})^*$. This unitary equivalence is effected by the matrix C: $C^{-1}\mathcal{D}^j(g)C = \mathcal{D}^j(g^{-1})^*$. We denote the matrices of this representation $\tilde{\mathcal{D}}^j(g)$ (so that $\tilde{\mathcal{D}}^j(g) = \mathcal{D}^j(g^{-1})^*$. The advantage of this representation is that for $g \in$ SU(2), we select in the equivalence class of (j,0) representations the representation $\mathcal{D}^j(g) = D^j(g)$, then $\tilde{\mathcal{D}}^j(g) = D^j(g)$ (whereas $\overline{\mathcal{D}^j(g)} = D^j(g^{-1})^t$); in other words, the representations $\mathcal{D}^j$ and $\tilde{\mathcal{D}}^j$ of SL(2,$\mathbb{C}$) coincide on its unitary subgroup.

In order to define this representation we begin once again

[5] The field Ψ^{-*} is simply related to the field Φ^+ by the matrix C: $\Psi^{-*} = C\Phi^+$; we have therefore, for a given one-particle Hilbert space, only two essentially differnt pairs of creation and annihiliation fields.

from the transformation properties of the annihilation operator Eq(19):

$$U_h a(k)^\sigma U_h = D^j(b(p)^{-1}h^{-1}b(hp))^{\sigma\sigma'} a(\delta(h).k)^{\sigma'}$$

We now expand the unitary matrix D^j in terms of the $(0,j)\sim$ representation:

$$D^j(b(p)^{-1}h^{-1}b(hp)) = \tilde{\mathcal{D}}^j(b(p)^{-1}\tilde{\mathcal{D}}^j(h^{-1})\tilde{\mathcal{D}}^j(b(h(p))$$

so that the annihilation operator

$$\tilde{a}^-(p)^\sigma = \tilde{\mathcal{D}}^j(b(p)^{\sigma\sigma'} a(p)^{\sigma'}$$

transforms as:

$$U_h \tilde{a}^-(p) U_h^{-1} = \tilde{\mathcal{D}}^j(h^{-1})\tilde{a}^-(\delta(h).p) = \mathcal{D}(h)^* \tilde{a}^-(\delta(h).p). \qquad (26)$$

As before, we define the creation operator

$$\tilde{a}^+(p)^\sigma = (\tilde{\mathcal{D}}^j(b(p))C^{-1})^{\sigma\sigma'} a^*(p)^{\sigma'}$$

which also transforms according to Eq.(26). We denote the corresponding annihilation and creation covariant spacetime fields $\tilde{\Phi}^-$ and $\tilde{\Phi}^+$ respectively. The transformation Eq.(26) is unitarily equivalent to the $(0,j)$ representation.

There is no covariant spacetime annihilation field which (anti)commutes with its adjoint; in this situation the canonical second quantization process of the non-relativistic theory appears to offer no guidance. We therefore adopt the strategy, to be justified *a posterori* by the consistency of the resulting formalism, that we must take linear combinations of the creation and annihilation fields. In this a further consideration immediately comes into force. Hitherto, complex numbers have entered into the theory in a linear or antilinear way; in particular the covariant creation (annihilation) fields transform linearly (antilinearly) when the underlying one-particle Hilbert space $\mathcal{H}$ is multiplied by a complex number of modulus one, i.e. under a rotation in each one-dimensional subspace (a global U(1) gauge transformation on $\mathcal{H}$). We here consider the fields as maps $\Phi : \mathcal{H} \longrightarrow L(\mathfrak{F}(\mathcal{H}))$, the set of linear operators on $\mathfrak{F}(\mathcal{H})$; the invariance of the expectation values of a self-adjoint operator on $\mathfrak{F}(\mathcal{H})$ under such gauge transformations of the space $\mathfrak{F}(\mathcal{H})$ is of course fundamental to the structure of quantum theory. By construction it also follows that the gauge transformations on $\mathcal{H}$ naturally induce

the gauge transformations of $\mathfrak{F}(\mathcal{H})$. Since for an arbitrary unitary transformation U of $\mathcal{H}$ one has

$\Gamma(UAU^{-1}) = \Gamma(U)\Gamma(A)\Gamma(U)^{-1}$

for any densely defined operator A on $\mathcal{H}$, and since for the creation and annihilation operators one has

$\Gamma(U)a(f)\Gamma(U)^{-1} = a(U^{-1}f)$,

we see that these two things are essentially the same; in particular, when $U:f \longrightarrow e^{i\theta}f$ it follows that $\Gamma(A) = \Sigma_{ij} a^*(f_i)(f_i, Af_j)a(f_j)$ (for f_i an orthonormal basis in $\mathcal{H}$) is gauge invariant.

We must now face up to the fact that this requirement is not compatible with the standard formalism of RQFT, unless some drastic revision is made in the canonical theory. There is no linear combination of creation and annihilation operators which transforms either linearly or antilinearly with the gauge transformations on the one-particle Hilbert space. Yet only such combinations have a chance of being causal. In this situation we shall not insist on the connection between the gauge transformations of the one-particle Hilbert space and transformations of the covariant fields; however we *shall* insist that nevertheless there is a U(1) global gauge group defined on the covariant fields, so that the observables of the theory can be defined as gauge invariant objects.

We formulate this requirement so that we may define *local* gauge invariant objects, or *at least one such.* That is, we demand: *there exists a global gauge group* $\zeta(x) \longrightarrow e^{i\theta}\zeta(x)$ *acting on the physical spacetime fields* ζ *and a self-adjoint bilinear form* g *on the spin space of the representation associated with* ζ *such that under the transformation* (h,a) *of the connected Lorentz group*

$$\zeta^*(x)^\tau g^{\tau\tau'}(x)\zeta(x)^{\tau'} \longrightarrow \zeta^*(x')g^{\tau\tau'}(x')\zeta(x')^{\tau'} \qquad (27)$$

where $x' = \delta(h).\ x+a$. It follows that $\zeta^* g\zeta$ is automatically guage invariant.

The qualification *physical* here means the covariant, causal fields; we do *not* assume that their associated spin representation is irreducible. We note that Eq.(27) is

satisfied when g is the unit in NRQFT. We shall call a field which satisfies these conditions **gauge covariant**. Henceforward the **physical field** satisfies covariance, causality, *and* gauge covariance.

Global U(1) gauge invariance clearly prohibits defining a causal covariant spacetime field as a linear combination of creation and annihilation operators acting on a *single* Hilbert space (of necessity, the annihilation operators will transform as $a \longrightarrow e^{-i\theta}a$ if the creation operator transforms as $a^* \longrightarrow e^{i\theta}a^*$). *We therefore introduce a second Hilbert space* $\mathcal{H}_b = L^2(\Theta^+, \mathcal{K}^J, \mu)$ (we denote the first Hilbert space $\mathcal{H}_a$ and refer to *a* and *b* particles accordingly). All of the foregoing is repeated for this second Hilbert space; we likewise denote the associated fields with subscript *a* and *b*. The type *b* particles are called **antiparticles**; they are described by the *same* irreducible unitary representation of the connected Lorentz group. The difference lies in the gauge transformations of Φ_a and Φ_b; we suppose that

$$\begin{aligned} \Phi_a^{\pm} &\longrightarrow e^{\pm i\theta}\Phi_a \\ \Phi_b^{\pm} &\longrightarrow e^{\mp i\theta}\Phi_b \end{aligned} \tag{28}$$

(by construction it then follows that the fields $\tilde{\Phi}$ transform in the same way; *e.g.* $\tilde{\Phi}_a^+ \longrightarrow e^{i\theta}\tilde{\Phi}_a^+$) so that linear combinations ζ of creation and annihilation operators in *a* and *b* respectively transform as $\zeta \longrightarrow e^{i\theta}\zeta$. We note that these tranformations of the covariant fields *cannot* be generated by gauge transformations of the underlying Hilbert spaces $\mathcal{H}_a$, $\mathcal{H}_b$ so long as the antiparticle space is *unitarily* equivalent to the particle space.

This point must be carefully examined. Intuitively, we expect that $\mathcal{H}_b$ can be associated with the representation on the negative mass hyperboloid, corresponding to the positive mass representation $\mathcal{H}_a$. Indeed, since there is a canonical anti-unitary correspondence $\mathfrak{C}$ between these representations (denote $\mathcal{H}_a^-$ and $\mathcal{H}_a^+$ respectively), then when an element of $\mathcal{H}_a^- \cup \mathcal{H}_a^+$ is subjected to the gauge transformation $f \longrightarrow e^{i\theta}f$, it follows that its restriction f^+ to $\mathcal{H}_a^+$ transforms as $f^+ \longrightarrow e^{i\theta}f^+$, whilst f^- under the correspondence $\mathfrak{C}$ transforms as $\mathfrak{C}f^- \longrightarrow e^{-i\theta}\mathfrak{C}f^-$. If we identify $f_a \in \mathcal{H}_a$ with f^+, and $f_b \in \mathcal{H}_b$

with $\mathfrak{C}\ f^-$ (so that $\mathcal{H}_b = \mathfrak{C}\ \mathcal{H}_a^-$) we will automatically ensure the correct gauge transformation properties of the creation and annihilation operators *so long as these operators are considered maps of the form:*

$$\Phi_a^{\pm}:\ \mathcal{H}_a^+ \longrightarrow \mathfrak{L}(\mathfrak{F}(\mathcal{H}_a^+))$$
$$\Phi_b^{\pm}:\ \mathcal{H}_a^- \longrightarrow \mathfrak{L}(\mathfrak{F}(\mathfrak{C}\ \mathcal{H}_a^-)) \tag{29}$$

(where $\mathfrak{L}(X)$ is the set of linear operators acting on the space X). The intervention of the anti-linear operator $\mathfrak{C}$ in the RHS of the second of these destroys the fundamental properties of the canonical second quantization. Section 3.4 is, in effect, a study of the modifications that must be made to the canonical second quantization in order to make sense of this.

$\mathfrak{C}$ may be taken as the *one-particle* charge conjugation operator introduced in (1.4.4). We defer discussion until (3.3.8) and (3.4.10). In the present context it may be any anti-isomorphism from $\mathcal{H}_a^-$ to $\mathcal{H}_a^+$.

If we cannot consider the transformations Eq.(28) as reflecting the underlying complex structure of the space $\mathcal{H}_a$, $\mathcal{H}_b$, then we must assume they are are associated with an independent complex structure of the fields themselves. This is consistent so long as Φ_a and Φ_b are not identical (that is, the particle and antiparticle are distinct). In the scalar case the identity of particle and antiparticle implies that the field is real; there is no gauge group. For non-integral spin the field is not real and if particle and antiparticle are identical again there is no gauge group. The proper interpretation of such theories, a theory of purely neutral fermions for example, will not be discussed here.

To see the relationship between antiparticle and causality we form the covariant field

$$\varphi(x) = \alpha\Phi_a^-(x) + \beta\Phi_b^+(x)$$

transforming via the $(\frac{1}{2},0)$ representation and evaluate its (anti) commutation relationships with the adjoint field. We find:

$$[\varphi(x)^{\sigma},\varphi^{*}(y)^{\sigma'}]_{\pm} = m^{-2j}(2\pi)^{-3}\cdot$$

$$\int \left\{\mathcal{D}^{j}(b(p))\mathcal{D}^{j*}(b(p))\right\}^{\sigma\sigma'}\left\{|\alpha|^{2}e^{-ip.(x-y)}\pm|\beta|^{2}e^{ip.(x-y)}\right\} d^{3}p/2p_{0}$$

In the case $j = 1/2$ we have $\mathcal{D}^{1/2}(b(p)) = b(p)$ and it is elementary to show that

$b(p)b(p)^{*} = (p_{0}\mathbb{1}+p_{i}\sigma^{i})/m$

(denote $p.\sigma$), where σ^{i} are the Pauli spin matrices, and hence that

$$[\varphi(x)^{\sigma},\varphi^{*}(y)^{\sigma'}]_{\pm} = im^{-1}\sigma^{\sigma\sigma'}.\frac{\partial}{\partial x}\left[|\alpha^{2}|\Delta^{+}(x-y) \mp |\beta^{2}|\Delta^{-}(x-y)\right].$$

If and only if we now set $|\alpha|^{2}=|\beta|^{2}$ and choose Fermi statistics, the term under the derivative becomes:

$\Delta^{+}(x-y) + \Delta^{-}(x-y) = \Delta(x-y)$

the causal delta function (Eq.(45)(1.4.4)). The anticommutator then vanishes for spacelike interval x-y; *the only causal field is that which contains both particle and antiparticle fields.* We have also obtained the connection between spin and statistics[6]. Indeed, this follows for arbitrary spin, since in the general case $\mathcal{D}^{j}(b(p)\mathcal{D}^{j}(b(p))^{*} = \mathcal{D}^{j}(b(p)^{2}) = \mathcal{D}^{j}(\sigma.p/m)$; this term may be written $\mathcal{D}^{j}(-im^{-1}\sigma.\frac{\partial}{\partial x})$ acting on the first exponential and $\mathcal{D}^{j}(im^{-1}\sigma.\frac{\partial}{\partial x})$ acting on the second, or

$$m^{-2j}(2\pi)^{-3}\mathcal{D}^{j}(i\sigma.\frac{\partial}{\partial x})\int\left[e^{-ip.(x-y)}\pm\mathcal{D}^{j}(-1)e^{ip.(x-y)}\right] d^{3}p/p_{0}.$$

Since $\mathcal{D}^{j}(-1) = (-1)^{2j}$ the (anti)commutator is causal for 2j (odd) even.

A similar result holds for the linear combination

$\chi(x) = \lambda\tilde{\Phi}^{-}_{b}(x) + \mu\tilde{\Phi}^{+}_{a}(x)$

provided once again $|\lambda|^{2} = |\mu|^{2}$; however if we also require that $\chi(x)$ commutes with $\psi(y)$ for spacelike x-y it is necessary to fix the relative phase of $\tilde{\Phi}^{-}$ and $\tilde{\Phi}^{+}$ as:

$\chi(x) = \tilde{\Phi}^{-}_{b}(x) + (-1)^{2j}\tilde{\Phi}^{+}_{a}(x)$.

[6] Weinberg developed the foregoing theory in the context of the general programme, of freeing the Feynman rules from any dependence on Lagrangian field theory, and recovering the empirically significant results of field theory (the TCP theorem, the spin-statistics theorem, analyticity properties, etc.) on the basis of the S-matrix programme. In this context this proof of the spin-statistics theorem should be contrasted with that of Stapp [1962], which made crucical use of analyticity postulates, but which was not restricted to free fields.

The necessity of considering *both* covariant, causal fields ψ and χ (transforming according to the $(j,0)$ and $(0,j)$-representations) becomes clear if we demand that there exists an invariant bilinear form g satisfying Eq.(27); for if this equation is satisfied by the physical field ζ transforming according to a (non-unitary) finite dimensional (not necessarily irreducible) representation S of the universal covering group of the inhomogeneous Lorentz group G^*, that is

$$U_{h,a}\zeta^{\tau}(x)U_{h,a}^{-1} = S(h^{-1})^{\tau\tau'}\zeta^{\tau'}(\delta(h)x+a)$$

then it is clear from Eq.(27) that $S(h^{-1})^* gS(h^{-1}) = g$ or

$$gS(h^{-1}) = S(h^{-1})^{*-1}g. \qquad (30)$$

Eq.(30) states that g is an intertwining operator for the representations S and S^{*-1} on the spin space associated with ζ. Choosing either one of $\zeta = \varphi$ or $\zeta = \chi$ clearly cannot satisfy this condition, since the representations $\mathcal{D}^j(h^{-1})$ and $\mathcal{D}^{j*}(h^{-1})^{-1}$ (for $\zeta=\varphi$) are unitarily inequivalent on $\mathbb{C}^{2j+1}$ (and similiarily for $\zeta=\chi$). But if we define the 2(2j+1)-component field

$$\zeta = \begin{bmatrix} \varphi \\ \chi \end{bmatrix}$$

which transforms as

$$U_h\zeta(x)U_h^{-1} = S(h^{-1})\zeta(\delta(h)x)$$

where S is the 2(2j+1) dimensional matrix

$$S(h) = \begin{bmatrix} \mathcal{D}^j(h) & 0 \\ 0 & \tilde{\mathcal{D}}^j(h) \end{bmatrix} = \begin{bmatrix} \mathcal{D}^j(h) & 0 \\ 0 & \mathcal{D}^{j*}(h)^{-1} \end{bmatrix}$$

acting on the spin space $\mathbb{C}^{2j+1} \otimes \mathbb{C}^{2j+1}$, then it is clear that the representations $S(h^{-1})$ and $S^*(h^{-1})^{-1}$ are indeed unitarily equivalent and that

$$\gamma^0 S(h^{-1})\gamma^0 = S^*(h), \quad \gamma^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

so that γ^0 is the unitary matrix which intertwines them. It follows that Eq.(27) is satisfied for $g = \gamma_0$ and that the bilinear form $\zeta^*(x)\gamma_0\zeta(x)$ transforms as a gauge invariant covariant scalar. As in Section 1.4 we denote $\zeta^*(x)\gamma^0$ as $\tilde{\zeta}(x)$, the covariant adjoint, transforming as

$$U_h\tilde{\zeta}(x)U_h^{-1} = \tilde{\zeta}(\delta(h)x)S(h);$$

the covariance of $\tilde{\zeta}\zeta$ is immediate.

We have motivated the essential innovation, that of

combining the (j,0) and (0,j)~ representations to form a reducible 2(2j+1) dimensional representation on the spin space of the quantum field, by an appeal to gauge invariance and the existence of a gauge invariant covariant scalar; we note that one may also appeal (Weinberg [1964]) to the requirement that it is possible to define a *charge conjugation operator* or a *parity operator* at the level of the spacetime fields[7]. Indeed, the parity inversion intertwines the (j,0) and the (0,j)~ representations, a well known result (see, e.g. Varadarajan [1970 p.181]). The present approach is due to Novozhilov [1975 5.1].

3.3.5. The Dirac field in standard form.

We elaborate on the description of the field $\zeta = \begin{bmatrix} \varphi \\ \chi \end{bmatrix}$ in the case j=1/2; ζ is then a 4-component field which may be written

$$\zeta(x)^{\tau} = (2\pi)^{-3/2}\int\left[u(p)^{\tau\sigma}a(p)^{\sigma}e^{-ip.x} + (v(p))^{\tau\sigma}a_b^{*}(p)^{\sigma}e^{ip.x}\right] d^3p/\sqrt{2p_0} \tag{31}$$

where we have defined u(p), v(p) as the 4×2 matrices

$$u(p) = \begin{bmatrix} \mathcal{D}^{1/2}(b(p)) \\ \tilde{\mathcal{D}}^{1/2}(b(p)) \end{bmatrix}, \qquad v(p) = \begin{bmatrix} \mathcal{D}^{1/2}(b(p))C^{-1} \\ -\tilde{\mathcal{D}}^{1/2}(b(p))C^{-1} \end{bmatrix}.$$

The similarity between (31) and the standard plane wave expansion of the Dirac field is obvious; the latter was written earlier (Eq.(35) (1.4.4))[8]:

$$\psi(x)^{\tau} = (2\pi)^{-3/2}\sum_{\sigma}\int\left[(u^{\tau}(p,\sigma)a_a(p,\sigma)e^{-ip.x} + v^{\tau}(p,\sigma)a_b^{*}(p,\sigma)e^{ip.x}\right]d^3p/\sqrt{2p_0} \tag{32}$$

where $u(p,\sigma)$ (respectively $v(p,\sigma)$) is a positive (negative) frequency Dirac bi-spinor satisfying

$$(\gamma^{\mu}p_{\mu} - m)u = 0, \quad (\gamma^{\mu}p_{\mu} + m)v = 0. \tag{33}$$

ψ transforms just as ζ, and admits the same bilinear invariant. Clearly we must have that $u^{\tau\sigma} = u^{\tau}(p,\sigma)$ and $v^{\tau\sigma} =$

[7] The existence of a parity operator does not require the existence of antiparticles, however; nor does the existence of a charge conjugation operator demand the existence of distinct antiparticles.

[8] With the minor change in notation: $b(p,r) \longrightarrow a_a(p,\sigma)$, $d(p,r) \longrightarrow a_b(p,\sigma)$ (the index σ = 1,2 replancing r = 1,2).

$v^{\tau}(p,\sigma)$; in establishing these identities, we shall see explicitly that the Eqs.(32) define constraints on the relationships between the coordinatizations of SU(2) and the reducible representation S of $SL(2,\mathbb{C})$ in the equivalence class of $(s,0)\otimes(0,s)\sim$.

As is clear from our notation, we assume from the outset that the creation and annihilation operators which appear in Eq.(32) are identical to those appearing in Eq.(31). This assumption is to be justified *a postiori* by the identity (up to a multiplicative constant) of ζ with ψ that is thereby established. With this assumption, we take over the analysis of (3.3.3) and conclude that the annihilation operators $a_a(p)$, $a_b(p)$ both transform as Eq.(19):

$$U_h a(p) U_h^{-1} = D^{1/2}(b(p)^{-1}h^{-1}b(\delta(h).p))a(\delta(h).p)$$

but now, when we expand the SU(2) matrix $D^{1/2}(\{h,p\})$ we use the reducible representation $S^{\tau\tau'}$ and correspondingly assume the existence of a linear transformation $\mathbb{T}$ connecting the basis used to coordinatize $\mathbb{C}^2$ in the canonical representation of SU(2) with the basis used to coordinatize the representation S; therefore $\mathbb{T}$ is a 4×2 matrix with components $\mathbb{T}^{\sigma\tau}$, where we assume $\mathbb{T}^{\sigma\tau}\mathbb{T}^{\tau\sigma'}=\delta^{\sigma\sigma'}$. We now write

$$D^{1/2}(\{h,p\})^{\sigma\sigma'} = \mathbb{T}^{\sigma\tau}S(b(p)^{-1})^{\tau\lambda}S(h^{-1})^{\lambda\mu}S(b(\delta(h).p))^{\mu\nu}\mathbb{T}^{\nu\sigma'}$$

from which it is clear that the operator

$$S(b(p))^{\tau\tau'}\mathbb{T}^{\tau'\sigma}a(p)^{\sigma}$$

transforms covariantly according to the representation $S(h^{-1})$. Comparsion with Eq.(32) leads one to conclude that

$$u^{\tau}(p,\sigma) = c\, S(b(p))^{\tau\tau'}\mathbb{T}^{\tau'\sigma} \qquad (34)$$

or that $u^{\tau}(\mathring{p},\sigma) = c\mathbb{T}^{\tau\sigma}$ (recall $\mathring{p} = (m,0,0,0)$), where c is a constant. The choice of the Dirac bispinor for the zero momentum state is therefore equivalent to a choice of the correspondence defined by $\mathbb{T}$.

The same analysis applied to the creation operator leads to a similar result; repeating the foregoing (but using Eq.(23)) we obtain the covariant creation operator:

$$S(b(p))^{\tau\tau'}\mathbb{T}^{,\tau'\sigma'}C^{-1\sigma'\sigma}a^{*}(p)^{\sigma} \qquad (35)$$

and the analogous expression

$$v(p,\sigma)^{\tau} = c\, S(b(p))^{\tau\tau'}\mathbb{T}^{,\tau'\sigma'}C^{-1\sigma'\sigma} \qquad (36)$$

which follows from a comparison of Eq(35) with Eq.(32).

Therefore $v^{\tau}(p,\sigma) = c\ \mathbb{T}'^{\tau\sigma'} C^{-1\sigma'\sigma}$.

If we write

$$\mathbb{T} = \begin{bmatrix} \mathbf{s} \\ \mathbf{t} \end{bmatrix},\ \mathbb{T}' = \begin{bmatrix} \mathbf{s}' \\ \mathbf{t}' \end{bmatrix}, \tag{37}$$

we note that $\mathbf{s},\mathbf{t}$, $\mathbf{s}',\mathbf{t}'$ are unitary, because the subrepresentations $\mathcal{D}^{1/2}(h^{-1})$, $\tilde{\mathcal{D}}^{1/2}(h^{-1})$, coincide when $h \in SU(2)$; however to proceed further will require an appeal to microcausality, and essentially a repeat of the arguments of (3.3.4) above. We observe rather that if we subject $\mathbb{T},\mathbb{T}'$ to the constraints

$$\mathbf{s} = \mathbf{t} = \mathbf{s}' = -\mathbf{t}' \tag{38}$$

we obtain

$$u(p) = c \begin{bmatrix} \mathcal{D}(b(p))\mathbf{t} \\ \tilde{\mathcal{D}}(b(p))\mathbf{t} \end{bmatrix},\ v(p) = c \begin{bmatrix} \mathcal{D}(b(p))\mathbf{t}C^{-1} \\ -\tilde{\mathcal{D}}(b(p))\mathbf{t}C^{-1} \end{bmatrix}$$

(using Eqs.(34),(14)) from which the correspondence of Eq.(32) with Eq.(33), which was to be proved, follows immediately (they are proportional, for $\mathbf{t}$ the identity; for any other choice, one simply effects a change in basis for $\mathbb{C}^2$).

The significance of Eq.(38) may be seen as follows; using the expressions for $u(p,\sigma)$, $v(p,\sigma)$ this choice may be imposed as the constraints:

$$\begin{aligned} \gamma^0 u(p,\sigma) &= u(p,\sigma) \\ \gamma^0 v(p,\sigma) &= -v(p,\sigma) \end{aligned} \tag{39}$$

We now substitute for the RHS using Eqs.(34),(36); the first of these becomes

$$\gamma^0 S^{-1}(b(p))u(p,\sigma) = S^{-1}(b(p))u(p,\sigma)$$

or, operating from the left by $S^{*-1}(b(p)) = S(b(p))$ and bearing in mind the invariance of γ^0 (Eq.(27)) the constraint:

$$S(b(p))\gamma^0 S^{-1}(b(p))u(p,\sigma) = u(p,\sigma)$$

and analogously for the second:

$$S(b(p))\gamma^0 S^{-1}(b(p))v(p,\sigma) = -v(p,\sigma)$$

Recalling that in the case $s = 1/2$

$$S(h) = \begin{bmatrix} h & 0 \\ 0 & h^{*-1} \end{bmatrix}$$

and that $b(p)b(p)^* = (\sigma^0 p_0 + \sigma^i p_i)/m$ one obtains respectively

$$\left\{\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}p_0 - \begin{pmatrix}0 & -\sigma^1\\ \sigma^1 & 0\end{pmatrix}p_1\right\} u(p,\sigma) = mu(p,\sigma)$$

$$\left\{\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}p_0 - \begin{pmatrix}0 & -\sigma^1\\ \sigma^1 & 0\end{pmatrix}p_1\right\} v(p,\sigma) = -mv(p,\sigma)$$

that is, Eqs.(33), the Dirac equations in momentum space, in the representation of Eq.(60) (1.4.4). To determine the constant c, we observe from the matrix expression for u and v that:

$$\sum_\sigma u(p)^\sigma \otimes \tilde{u}(p)^\sigma = c^2\left[\begin{array}{c|c} b(p)b(p)^{-1} & b(p)b^*(p) \\ \hline b^*(p)^{-1}b(p)^{-1} & b(p)^{-1}b(p)\end{array}\right]$$

$$= \frac{c^2}{m}\left[\begin{array}{c|c} \mathbb{I} & \sigma^0 p_0 + \sigma^1 p_1 \\ \hline \sigma^0 p_0 - \sigma^1 p_1 & \mathbb{I}\end{array}\right] = \frac{c^2}{m}(\gamma^\mu p_\mu + 1)$$

$$\sum_\sigma v(p)^\sigma \otimes \tilde{v}(p)^\sigma = \frac{c^2}{m}\left[\begin{array}{c|c} -b(p)C^{-1}Cb(p)^{-1} & b(p)C^{-1}Cb^*(p) \\ \hline -b^*(p)^{-1}C^{-1}(-C)b(p)^{-1} & C(-b(p)^{-1})b(p)C^{-1}\end{array}\right]$$

$$= \frac{c^2}{m}\left[\begin{array}{c|c} -\mathbb{I} & \sigma^0 p_0 + \sigma^1 p_1 \\ \hline \sigma^0 p_0 - \sigma^1 p_1 & -\mathbb{I}\end{array}\right] = \frac{c^2}{m}(\gamma^\mu p_\mu - 1)$$

whence in comparison to Eq.(43) (1.4.4) we see that $c^2=m$.

Apart from constraining its solutions to the two-sheeted mass hyperboloid, we see that the Dirac equation is simply the covariant form of the constraint (Eq.(38) or (39)) which singles out a 2-dimensional subspace of the spin space on which the representation $S(h^{-1})$ acts. We also see that this constraint is automatically taken into account with the expansion of the covariant causal field in Eq.(31), while its expansion in Eq.(32) must be supplemented with these constraints. And finally, we have for the first time established a connection between anti-particle states and the *negative frequency* solutions of the Dirac equation. The *creation* part of the covariant causal field acts on the vacuum to produce an antiparticle, the spin components of which are constrained in the same way as a negative frequency plane wave solution of the Dirac equation. This is established on the postulates of covariance, local commutativity, and gauge covariance, proceeding from the

canonical unitary representations of the spin half particle. However, by construction, the state in $\mathfrak{F}(\mathcal{H}_b)$ thus produced transforms by the *positive* mass hyperboloid spin 1/2 Wigner representation. This puzzling situation will not be resolved until Section 3.4.

3.3.6. Representations on Hilbert space bundles.

The basic idea of the Weinberg construction - the factorization of the cocycles which define the canonical representation and the assimilation of one such factor to the states themselves - can be understood in a geometrically intuitive way. This insight is not so deeply hidden in the consideration, evident from the inception of the Dirac theory, that although the spin representation is non-unitary with respect to the canonical inner product on $\mathbb{C}^4$ it is nevertheless unitary with respect to the inner product $\int \Sigma \tilde{\psi}(x)\gamma_\mu \psi(x)\, d\sigma^\mu(x)$ on the solution manifold M (the complex linear space of configuration space solutions of the Dirac equation); see the discussion of (1.4.2) (Eq.(15) *et seq*). In some sense the inner product itself transforms in order to maintain unitarity, or one has a family of inner products (one for each spacelike hypersurface) and, whilst the spin representation is non-unitary with respect to any one of these, it is a unitary transformation from one inner product space to another. Let us now express these ideas clearly.

We proceed from the canonical theory of (2.4.6); for X a transitive G-space, and G_0 the stability subgroup at $x_0 \in X$, we suppose we are given an invariant measure μ and a unitary representation m of G_0 on a separable finite-dimensional Hilbert space $\mathcal{K}$. So far this is the basis (but with *irreducible* m), with $X = \Theta^+$, from which we uniquely define an equivalence class of irreducible unitary representations of G via the inducing construction. We now make a further assumption; that *there exists a finite-dimensional Hilbert space $\mathcal{K}'$ and a homomorphism* m′ *of G into the group of invertible operators of $\mathcal{K}'$ such that:*

(i) $\mathcal{K}$ is a closed linear manifold of $\mathcal{K}'$

(ii) $m'(g)w = m(g)w$ for all $g \in G_0$ and $w \in \mathcal{K}$.

It is clear that this is the situation that we arrived at via the Weinberg construction; m' corresponds to the finite dimensional non-unitary representation $\mathcal{D}^j$ of $SL(2,\mathbb{C})$, m to the unitary representation of $G_0 = SU(2)$, $\mathcal{K} = \mathbb{C}^{2j+1}$, $\mathcal{K}' = \mathbb{C}^{2j+1}$, and the condition (ii) to the fact that $\mathcal{D}^j$ restricted to the rotation subgroup coincides with the canonical representation of $SU(2)$.

We now define a *family* of Hilbert spaces $\mathcal{K}_x = m'(g)\mathcal{K}$ where $g.x_0=x$ (such a g must exist because X is transitive by assumption; note that it is *not* unique). On each $\mathcal{K}_x$ there exists an inner product $\langle .,. \rangle_x$ defined by:

$$\langle v,w \rangle_x = \langle m'(g^{-1})v, m'(g^{-1})w \rangle \text{ for } v,w \in \mathcal{K}_x$$

where $\langle .,. \rangle$ is the inner product on $\mathcal{K}$. It is obvious that

$$\mathcal{K}_{g.x} = m'(g)\mathcal{K}_x$$

for *all* $g \in G$, $x \in X$ (because m' is a homomorphism of G) and that

$$\langle m'(g)v, m'(g)w \rangle_{g.x} = \langle v,w \rangle_x .$$

We conclude that $m'(g)$ is an isometric bijection of $\mathcal{K}_x$ onto $\mathcal{K}_{g.x}$ and is hence *unitary*. Now introduce the Hilbert space of $\mathcal{K}'$ valued square integrable functions on X (with respect to the measure μ), that is, the set of all $\varphi: X \longrightarrow \mathcal{K}'$ such that

$$\|\varphi\| = \int_X \langle \varphi(x), \varphi(x) \rangle_x \, d\mu(x)$$

is finite. This space, with inner product derived from this norm, we denote $H^{X,\mathcal{K}'}$. Observe that for any section c (that is, a map from X into G such that $c(x).x_0 = x$) it follows that

$$\langle \varphi(x), \psi(x) \rangle_x = \langle m'(c(x)^{-1})\varphi(x),\ m'(c(x)^{-1})\psi(x) \rangle \qquad (40)$$

The elements $m'(c(x)^{-1})\varphi(x) \in \mathcal{K}$ (for fixed x) correspond to the vectors in $\mathbb{C}^{2j+1}$ defined by the generalized eigenfunctions $f_q(p)$ of (3.3.3), and the $\varphi(x)$ as vectors in $\mathcal{K}_x$ correspond to the vectors in $\mathbb{C}^{2j+1}$ defined by the covariant generalized eigenfuntions produced by application of $a^{-*}(q)$ to the vacuum, that is generalized eigenfunctions of the form $\mathcal{D}^S(b(q))f_q(p)$. In particular the "boosts" $b(q)$, which we defined earlier as the hermitian elements of $SL(2,\mathbb{C})$, such that $b(q)\overset{\circ}{p} = q$, constitute a particular choice

of section c. We see that the family of inner products $\langle .,. \rangle_x$ are defined precisely so as to agree, with respect to the covariant states, with the inner product of the associated canonical states $m'(c^{-1}(x))\varphi(x)$.

In fact the map Π: $H^{X,\mathcal{K}'} \longrightarrow \mathcal{H} = L^2(X,\mathcal{K},\mu)$ defined by
$(\Pi\varphi)(x) = m'(c(x)^{-1})\varphi(x)$
is an isometric bijection, since by Eq.(40) we see that
$\langle \varphi(x),\varphi(x)\rangle_x = \langle (\Pi\varphi)(x),(\Pi\varphi)(x)\rangle$.
Consider now the canonical system of imprimitivity on $\mathcal{H}$ defined by
$(P_E f)(x) = \chi_E(x)f(x)$
$(U_g f)(x) = \phi(g,g^{-1}.x)f(g^{-1}.x)$
for $f \in \mathcal{H}$ (*cf.* Eq.(13)), induced by the representation m of G_0 at x_0. We now define the covariant system of imprimitivity as the pair $P'_E = \Pi^{-1}P_E\Pi$, $U'_g = \Pi^{-1}U_g\Pi$ on $H^{X,\mathcal{K}'}$. Both systems are based on X and it is obvious that $P' = P$. ϕ is a (G,X,M) cocycle with values in M, the unitary group of $\mathcal{K}$, satisfying $\phi(g,x_0) = m(g)$ for $g \in G_0$. We verify that this relationship is satisfied by

$$\phi(g,x) = m(c(g.x)^{-1}gc(x)), \tag{41}$$

(note that $c(g.x)^{-1}gc(x) \in G_0$, so the RHS is well-defined), as are the cocycle identities Eq.(2) (2.4.6), for an arbitrary section c. By assumptions (i), (ii) above we conclude that
$\phi(g,x) = m'(c(g.x)^{-1}gc(x))$
defines the same representation as Eq.(41) and we use this cocycle. The representation U'_g is then defined by
$(U'_g\varphi)(x) = m'(c(x))m'(c(x)^{-1}gc(g^{-1}.x))f(g^{-1}.x) = m'(g)\varphi(g^{-1}.x)$.
The factoring of the representation m using the representation m' is of course familiar from (3.3.4). There is, however, the difference that we there obtained (Eq.(22)) the transformation law with g^{-1} in place of g[9].

[9] One obtains the reverse action on the generalized eigenstates; the group action is switched to the action on the generalized eigenvalues rather than on the arguements of the states as functions on X. The difference in the argument of the m' and $\mathcal{D}^j$ representations is conventional, the consequence of defining the covariant annihilation operator so that it transforms by multiplication from the left by the

We have proved the following theorem:

Theorem 3.3.1.

Given the assumptions (i),(ii), and $H^{X,\mathcal{K}'}$, P', U' as defined above; then (P',U') is a system of imprimitivity on $H^{X,\mathcal{K}'}$, and its equivalence class is the one induced by the representation m of G_0. It is irreducible if and only if m is irreducible.

The last statement is a consequency of (Th.2.4.3). One can also show (Varadarajan [1970 p.85]) that the assumptions (i), (ii) are always satisfied for the Lorentz group if G_0 is maximal compact in G (that is, no compact subgroup of G properly contains G_0), and that in this case the extension of m by m' is actually *unique*.

The consequences of assumptions (i) and (ii), which are sufficient for the proof of Theorem (3.3.1), may also be obtained from the following assumptions: there exists G-spaces X, B, such that X is transitive, and there exists a map π: B $\longrightarrow$ X such that $\pi(D(g)b) = g.\pi(b)$ (b $\in$ B, where D(g) is the automorphism of B induced by g). Further, for each x $\in$ X, $B_x = \pi^{-1}(x)$ is a separable Hilbert space and b $\longrightarrow$ D(g)b is a unitary isomorphism of B_x onto $B_{g.x}$, where b $\in B_x$, for all (g,x) $\in$ G $\times$ X. B is then called a **G - Hilbert space bundle** (a vector bundle over X). A **cross-section** of B is a map f: X $\longrightarrow$ B such that $f(x) \in B_x$ for all x $\in$ X; clearly the Hilbert space $H^{X,\mathcal{K}'}$ constructed above is the Hilbert space of square integrable cross-sections of the G-Hilbert space bundle X $\times$ $\mathcal{K}'$. More generally, for a G-Hilbert space bundle B we denote the asociated Hilbert space of square integrable cross sections of B by H^B. We thus have a precise geometrical interpretation of the covariant spinor states which underly the covariant second quantization discussed above.

matrix representation $\mathcal{D}^s$. These technicalities are unfortunate but appear to be unvoidable if one wants to define the standard plane-wave expansion for the covariant fields.

We conclude with some heuristic comments concerning the Dirac representation. From the present viewpoint this representation derives from the assumption, not that the extension m′ of m is given by the $\mathcal{D}^{1/2}$ representation, but rather by the reducible representation $S = \mathcal{D}^{1/2}\otimes\tilde{\mathcal{D}}^{1/2}$. To meet the assumption (i), (ii) above it is necessary to define a *two*-dimensional fibre $\pi^{-1}(p)$ over each $p \in \Theta^{+}$, as a subspace of $\mathbb{C}^4$ closed and stable under $S(g)$, $g \in G_p$, such that $\pi^{-1}(p)$ is mapped onto $\pi^{-1}(p')$ by $S(g)$ when $\delta(g).p = p'$.

It is a remarkable property of the representation S of $SL(2,\mathbb{C})$ on $\mathbb{C}^4$ that this can be done; if we define

$\pi^{-1}(p) = \{u \in \mathbb{C}^4;\ (p_0\gamma^0-p_1\gamma^1-p_2\gamma^2-p_3\gamma^3)u = mu\ \}$

then this constraint is preserved by the group action

$(p,u) \longrightarrow (\delta(g).p,\ S(g)u)$

hence both conditions are satisfied; the constraint moreover has a local form in configuration space, or rather it would do if it were not formulated on Θ^{+} (it is the Dirac equation restricted to positive frequencies). The bundle $\{p,\pi^{-1}(p)\} = B^{\Theta^{+},\mathbb{C}^2}$ so defined can be considered an orbit in $B^{\mathbb{P}^4,\mathbb{C}^4}$: the KG equation picks out the orbit $B^{\Theta^{+}\cup\Theta^{-},\mathbb{C}^4}$, the restriction to the positive mass hyperboloid picks out $B^{\Theta^{+},\mathbb{C}^4}$, and the Dirac equation picks out the subspace $\pi^{-1}(p)$ of $\mathbb{C}^4$ as a 2-(complex) dimensional fibre. We now see that the Dirac representation is in fact irreducible even though the representation S is not.

Similar remarks apply to the representation over the negative mass hyperboloid. The constraint is exactly the same, except that p is restricted to Θ^{-}. Therefore one can use the single constraint, for arbitrary p, that $(p_0\gamma^0-p_1\gamma^1-p_2\gamma^2-p_3\gamma^3)u = mu$ (it then follows that $p \in \Theta^{+}\cup\Theta^{-}$); this single constraint is the Dirac equation. In the position space representation, where one has the time-dependent wave functions $f^{\pm}_p(t) = e^{\mp iHt}f^{\pm}_p$, $f^{\pm}_p \in L^2(\mathbb{P}^3,\mathbb{C}^2,d^3p)$, one can think of the f_p's as being subject to *different* constraints (corresponding to the two choices of sign). As a result, passing from this representation on the

two-sheeted hyperboloid to the Dirac representation via the Foldy-Wouthuysen transformation:

$$\begin{bmatrix} f^+_p \\ f^-_p \end{bmatrix} \text{ —— } F^{-1} \longrightarrow \varphi \in B^{\Theta^+ \cup \Theta^-, \mathbb{C}^2}$$

it might appear that a single constraint on φ now replaces the constraints on $f^{\pm}$, and that φ has 4 components in order for this to be possible. But we could equally well write:

$$\begin{bmatrix} f^+_p \\ f^-_p \end{bmatrix} \text{ —— } F^{-1} \longrightarrow \begin{bmatrix} \varphi^+ \\ \varphi^- \end{bmatrix}, \quad \varphi^+ \in B^{\Theta^+, \mathbb{C}^2}, \ \varphi^- \in B^{\Theta^-, \mathbb{C}^2}$$

and the possibility of combining φ^+ and φ^- as a coherent superposition only appears natural because the constraint for each is the same, arising from a single configuration space wave equation.

In the Weinberg construction we were able to motivate the combination of the $(j,0)$ and $(0,j)$~ representations, which defines the representation S, by appeal to gauge invariance or to parity. Evidently only the latter is available from the point of view of group theory. But from a purely mathematical point of view there are other considerations which make this representation a very special one. It is, in fact, exactly as special as the 4-dimensional Clifford algebra, because it is the *unique* representation of the HLG as a group of automorphisms on this algebra.

I shall not define the Clifford algebra abstractly[10], but it is a C^*-algebra, and finding a representation for the infinite-dimensional Clifford algebra does much the same thing as finding a representation for the infinite-dimensional Weyl algebra: it defines a quantum field theory (see (3.4.5)). In finite dimensions the Clifford algebra is in some sense the fermionic version of a symplectic geometry. The remarkable properties of the Dirac equation have long exercised comment (*cf.* Wightman [1972]); the Weinberg construction does not get to the bottom of it.

[10] For a heuristic discussion see fn. 12, (1.4.2).

3.3.7 Canonical second quantization; position space.

We recall that the canonical second quantization of the non-relativistic quantum mechanics (NRQM) led to a remarkable correspondence between the formal expressions for the expectation values of one-particle operators (as configuration space integrals) and the global operators for the quantum field. In effect, one has only to replace the configuration space states (as complex valued functions on $\mathbb{R}^3$) by quantum fields (operator valued functions on $\mathbb{R}^3$) obeying appropriate commutation relationships.

We recall that Dirac's paper of [1927a] formulated two techniques[11] for generating the free non-relativistic quantum field (methods 2 and 3 respectively in the terminology of Sections 1.2, 1.3):

A: The use of creation and annihilation operators, beginning from a particle ensemble of arbitrary number obeying Bose statistics and described by NRQM (canonical second quantization).

B: A canonical quantization of a classical Lagrangian linear field theory, with particle structure defined by the simple interpretation: the canonically conjugate fields are creation and annihilation operators.

Both approaches led to the same formalism; *in addition*, it was evident that a third possibility existed:

C: Write down the configuration space expressions for the one-particle expectation values; replace the c-number functions $\psi(\mathbf{x})$ by q-number functions $\Psi(x)$ satisfying the commutation relationships $[\Psi(\mathbf{x}),\Psi^*(\mathbf{x}')] = i\hbar\delta^3(\mathbf{x}-\mathbf{x}')$.

The latter we called the *local equivalence* of field and one particle theory. As we stressed in (1.3.3), the existence

[11] And also a third, his "method 1", which we are not concerned with here.

of this local equivalence is of some interest from the point of view of the wave-particle duality, in that one can replace the Born interpretation by the Schrödinger interpretation (of course, for this to be possible the function $\psi^*(\mathbf{x})\psi(x)$ must admit a Born interpretation).

We summarize the conclusions of Section 1.4: by 1934, there existed a modified version of (B) (modified, that is, by the supplement of a new particle interpretation, defined by the plane wave expansion), and the normal-ordered global operators for energy, momentum, particle number and charge all took on the usual form under this particle interpretation, under the assumption that the antiparticles were of positive energy and opposite charge. This was, further, in agreement (in the spin 1/2 case) with a modified version of A, modified that is through the Dirac hole theory. I know of no attempt to formulate a correspondence with B via the *canonical* second quantization of a pair of 1-particle theories, (*e.g.* as formulated above); in the absence of detailed knowledge about the representation theory of the ILG the nearest thing to a particle theory with which to try to construct a cannonical version of A was provided by the positive frequency solutions to the free covariant wave equations. If one tries to do this, one will not obtain the annihilation and creation parts of the standard formalism, except for the scalar field. One obtains annihilation operators $a(\varphi_a^+)$, $a(\varphi_b^+)$ of the form Eq.(10), (this is what we shall do in Section 3.4), but there is no way to pass from these to the operators $u(p,\sigma)a_a(p,\sigma)$, $\overline{v(p,\sigma)}a_b(p,\sigma)$, by use of generalized eigenstate solutions in place of φ^+. The relationship between these annihilation operators (the $a(\varphi_a^+)$'s and the $a_a(p,\sigma)$'s) is more subtle, because they act on Hilbert spaces carrying different representations, as we have just seen. Without a doubt there existed a logical hiatus between A and B prior to the early 60's.

This hiatus was masked by the tendancy to simultaneously interpret the Lagrangian theory as the quantum mechanics of the relativistic particle (at least in the free case). Of

course, just such a dual interpretation underlies C in the non-relativistic case. In a sense which was never clearly defined, the negative frequency states were interpreted as anti-particle states. However the formal prescription (and we are now looking at a variant of C) was unambiguous: all that was necessary was to re-interpret the 1-particle inner product as the total charge operator (rather than the number operator), adopt the correct plane wave expansion, and normal order everything in sight. In the Dirac case one had the even closer correspondence with the 1-particle theory provided by the hole theory; one could take over the plane wave expansion for the one-particle states of $B^{\Theta^+\cup\Theta^-,\mathbb{C}^2}$ (Eq.(22) of (1.4.4)):

$$\varphi(x) = 2^{-1/2}(2\pi)^{-3/2}\int_{\Theta^+\cup\Theta^-}\Sigma_\sigma u(p,\sigma)a(p,\sigma)e^{-ip.x}d\mu \qquad (42)$$

and suppose that the prescription "make $\varphi(x)$ an annihilation operator" means "make $a(p,\sigma)$ an annihilation operator". The hole theory then takes one from Eq.(42) to the plane wave expansion of the standard formalism (the transition from Eq.(22) to Eq.(35) of (1.4.4)).

We have provided a bridge between A and B, but C, the local equivalence, remains as impenetrable as ever. The best that we can do is to write down the densities, as bilinear expressions in the creation and annihilation operators of the particles and antiparticles separately, and verify their formal agreement with the (naïve) 1-particle positive frequency configuration space probability distributions. In this way, for example, the momentum density operator

$$\psi_a^-(x)^*(-i\hbar\gamma^1\frac{\partial}{\partial x^1})\psi_a^-(x) + \psi_b^-(x)^*(-i\hbar\gamma^1\frac{\partial}{\partial x^1})\psi_b^-(x) \qquad (43)$$

takes the same form as the (naïve) particle-antiparticle probability density. (We have written $\psi_a^- = \begin{bmatrix}\Phi_a^- \\ \tilde{\Phi}_a^-\end{bmatrix}$, $\psi_b^- = \begin{bmatrix}\Phi_b^{+*} \\ \tilde{\Phi}_b^{+*}\end{bmatrix}$, the physical particle and antiparticle annihilation fields.) These fields do *not* commute at spacelike separations; the configuration space wave functions (of positive frequency) do *not* support a Born interpretation; and finally, the one-particle analogues of field denisities will only exist when the associated 1-particle configuration space observable is even, in the sense of Schrödinger.

What we have not captured, is the way in which the negative frequency solutions can be associated with antiparticle states. We know that we can consider the positive frequency antiparticle states, elements of $\mathcal{H}_b^+$, as the charge conjuates of the elements of the negative frequency particle states, lying in $\mathcal{H}_a^-$. But, to repeat our earlier discussion, we do not know how to incorporate this way of thinking about type b particles into the formalism, because we must make sense of the idea that the annihilation and creation operators must be defined as maps of the form Ψ: $\mathcal{H}_a^+ \cup \mathcal{H}_a^- \longrightarrow \mathcal{L}(\mathfrak{F}(\mathcal{H}_a^+) \otimes \mathfrak{F}(\mathcal{H}_b^+))$. We need a change at a foundational level of the canonical second quantization. We have already indicated the necessary background (2.5.7); this is applied to the present context in the next section.

To return to the local equivalence, whilst we are resigned to working separately with the creation and annihilation fields for each particle type (i.e. as in Eq.(43)), we can at least eliminate the problems detailed above by working from a pair of 1-particle theories in the *position space* representation. That is, we must recognize that from the point of view of effecting a correspondence between the particle theory and the field which preserves the Born interpretation, we must define a *q-local correspondence.*

In position space, the associated creation and annihilation fields will be causal (we shall say *q-causal*), as follows from the simple observation that the equal time CCR
$[a(f_1), a^*(f_2)] = (f_1, f_2)$
will be zero when the supports of position space states f_1, f_2 are disjoint, only when the inner product is then zero. But this is exactly the situation (*cf.* (3.2.7), (3.2.8)) that characterizes position space and guarantees a Born interpretation; the momentum space representation (position space dual) has Hilbert space $L^2(\mathbb{P}^3, \mathbb{C}^{2j+1}, d^3p)$ which is isomorphic under τ^+ ((3.2.6) Eq.(22)) to the Wigner representation on $L^2(\Theta^+, \mathbb{C}^{2j+1}, d\mu^+)$, and the position space Hilbert space is $L^2(\mathbb{R}^3, \mathbb{C}^{2j+1}, dx^3)$. Therefore the generalized eigenstates of postion are of the form

$f^{\sigma}_{q}(x) = w\,\delta^3(q-x)$

where w is a $2j+1$ dimensional normalized spinor and the creation and annihilation operators $a(q,\sigma) = a(f^{\sigma}_{q})$, etc. for these states obey the equal time CCR:

$[a(q,\sigma), a(q',\sigma')^*] = \delta^3(q-q')\delta^{\sigma\sigma'}$

They are therefore causal. We therefore have a q-local correspondence between bilinear field densities and the associated one-particle probability densities:

$a^*_a(q)O_a(q)a_a(q) + a^*_b(q)O_b(q)a_b(q) \longleftrightarrow \overline{f_a(q)}O_a(q)f_a(q) + \overline{f_b(q)}O_b(q)f_b(q)$

in the sense of (1.3.3) for densely defined q-local one particle operators $O(q)$ (i.e. multiplicative functions of q or finite derivatives in q). However we know that q-local operators are not c-local; neither the q-local number density operator (defined as above for $O_a(q) = O_b(q) = 1$) nor the q-local charge density operator (defined by $O_a(q) = -O_b(q) = -e$) are c-local[12]. We have no commuting c-local number density operator in configuration space, of course, for the same reason that we do not have a Born interpretation. However we do have a commuting c-local charge density operator, which is moreover what is actually coupled to the electromagnetic field; this charge density is not q-local. However the total charge and number operators, defined by integration of the densities over position space or configuration space, are in agreement. The truth of these assertions is simply checked, and may be found (in the scalar case) in Schweber [1961 7c].

[12] From a purely particle point of view, there is no theoretical motivation for defining this q-local charge operator. In the present context the charge operator only arises in a natural way from the Lagrangian theory, as the generator of the gauge transformations of the Lagrangian, in the well-known way.

3.3.8. Conclusions.

We see that there is a formulation of C which better reflects our original interest, that the probability density for an ensemble of particles is equal to the expectation value of the number density operator. We have a q-local correspondence. In a very direct sense measurement of the "wave-like" properties of a particle beam (which always involve measurements of particle density) is the same thing as the measurement of the expectation value of a q-local bilinear form in the underlying position space quantum fields. Once again we have a fundamental dichotomy between covariant c-local observables, constructed as local expressions (in configuration space) in the physical fields, and q-local observables, constructed as local expressions (in position space) in the q-local fields. Which, if any, is the "real" local quantum field? We have already addressed the 1-particle analogue of this question in Section 3.2. One must surely respond, that the covariant c-local observables which actually are locally coupled to other (covariant) c-local fields to define interactions must be considered primary from a dynamical point of view. On the other hand, from an epistomological point of view, spacetime events are defined with respect to the localization of particles, hence q-locality; if, in some sense, one could probe the interaction region with particles one would conclude *with respect to particle locality* (hence q-locality) that the interaction is non-local (for so it would appear in position space). We cannot do this in a phenomenological context; we draw this conclusion from the perspective on spacetime offered by the concept of particle locality. From *this* perspective, the covariant fields are a purely metaphysical construction; the spacetime properties of these fields are only known when one defines corresponding fields on position space (q-local fields). The position space fields are what the covariant c-local fields look like from a particle perspective, and they are just as real, or just as useful, as the particle concept is real or useful in interpreting the meaning of locality, which leads to q-locality.

To return to the physical quantum fields, we have seen that they are essentially configuration space objects constructed from linear combinations of canonical creation and annihilation operators over a pair of unitarily equivalent Fock spaces, but with different properties under gauge transformations. It is for the latter reason that the two must be distinct. The gauge transformation properties follow if we regard these Fock spaces as the second quantization of a positive mass-hyperboloid representation, and of the charge conjugate of the corresponding negative mass-hyperboloid representation. But we cannot do this; the antiparticle states are *unitarily equivalent* to the particle states. It is as if we have *two* different complex structures present, the one determining the gauge transformations, the other the complex structure of the group representation theory, the complex numbers which enter into the CCR's. As we shall see in Section 3.4, this is indeed the case.

Further, we have obtained no clue as to why the demands of microcausality enter so critically into the specification of the particle interpretation of the physical fields. Once again, we refer to Section 3.4 for the resolution of these difficulties.

One last point deserves mention. The charge conjugate field, defined by the interchange of the particle and antiparticle fields, is described at the level of the physical fields by the same formal transformation as the charge conjugation in the one-particle theory. We recall (1.4.4) that the charge conjugate of a negative frequency solution of the Dirac equation (coupled to a fixed electromagnetic field) is a positive frequency solution of the same equation, but with the sign of the electromagnetic potentials reversed; that is, for an arbitrary solution ϕ of the Dirac equation,
$(i\gamma^{\mu}\partial_{\mu} - mc)\phi = e\gamma^{\mu}A_{\mu}\phi$
for a potential A_{μ}, then $\phi^{c} = \mathcal{C}\phi = \mathcal{C}\bar{\phi}^{t}$ satisfies:
$(i\gamma^{\mu}\partial - mc)\phi^{c} = -e\gamma^{\mu}A\ \phi^{c}$
and if $\phi \in H(B^{\Theta^{\pm}}, \mathbb{C}^{2})$ then $\phi^{c} \in H(B^{\Theta^{\mp}}, \mathbb{C}^{2})$ (The unitary matrix $\mathcal{C}$ is defined by Eq.(62)(1.4.4.); it also follows from the

Weinberg construction that it is the matrix $\begin{pmatrix} C & 0 \\ 0 & -C \end{pmatrix}$, where C is defined as in (3.3.3) (and can be taken equal to the matrix C given explicitly in Eq.(62) (1.4.4)). From Eq.(32) or (33), it is elementary to prove, that the same formal transformation at the level of the physical quantum field (which we denoted $\mathfrak{C}_F$), that is:

$$\mathfrak{C}_F: \psi \longrightarrow \mathfrak{C}\tilde{\psi}^t \qquad (44)$$

is equivalent to the transformation (*cf.* Eq.(64)(1.4.4):

$$a_a(k) \longleftrightarrow a_b(k). \qquad (45)$$

We therefore have the peculiar situation, that we have remarked upon in (1.4.4), that whereas the transformation Eq.(44) appears anti-linear, that of Eq.(45) is linear. Just because $\mathcal{H}_a$ and $\mathcal{H}_b$ are unitarily equivalent, $\mathfrak{C}_F$ acting on the physical fields *cannot* be the canonical second quantization of $\mathfrak{C}$ acting on $\mathcal{H}_a$ and $\mathcal{H}_b$.

There is one further strand to the peculiar properties of charge conjugation in the Dirac theory; the one-particle charge conjugation does *not* reverse the sign of the charge-current 4-vector:

$\mathfrak{C}$: $e\phi(x)\sim\gamma_\mu\phi(x) \longrightarrow e\phi(x)\sim\gamma_\mu\phi(x) = e\,\phi^{c}(x)^{\sim}\gamma_\mu\,\phi^{c}(x)$.

(where $\phi\sim$ is the covariant adjoint). Indeed it cannot, since its time component, the charge density, is positive definite, which is why (divided by e) it was taken to define the probability density. But at the level of the fields $\mathfrak{C}_F$ indeed reverses the sign of this quantity, because taking the adjoint reverses the order of the creation and annihilation operators; the charge -current operator is normal ordered, and this introduces a change in sign. In the scalar theory the normal ordering does not change the sign - and indeed at the one-particle level the sign is reversed also. These peculiarities have led to the majority view that there is no consistent charge conjugation operator at the 1-particle level[13]; I say rather that we have one more mystery. What is the relationship between $\mathfrak{C}_F$ and $\mathfrak{C}$? This problem too is resolved in the following section.

[13] e.g. Pais [1986 p.381]: "the great novely about C-invariance is that it can only be consistently formulated in a quantized field theory."

3.4. Geometric Field Quantization.

> The negative energy states *had* to be taken seriously.....
>
> P. Dirac

3.4.1. Introduction.

In Section 3.1 - 3.3 we examined the structure of free field theory form a purely group-theoretic basis. New elements intrude, microcausality, and gauge covariance, which are foreign to the canonical second quantization of NRQM. Without gauge covariance, we naturally obtain covariant causal fields, with the particle interpretation of the real scalar field; no antiparticles, no pair production, and no conserved charge.

In the canonical second quantization, the commutation relationships of the fields are determined by those for the canonical creation and annihilation operators. These in turn are determined by the Hilbert space, the second quantization of which is the Fock space for the fields. As is made clear in the position space representation for the fields, these can be made to assume a local form: for a spin zero particle the one-particle space is $\mathcal{H} = L^2(\mathbb{R}^3, d^3q)$ and we find for the equal time CCR:

$$[a(f), a^*(f')] = (f,f')_{\mathcal{H}} = \int \overline{f(\mathbf{q})} f'(\mathbf{q}) d^3q$$

so that the commutator vanishes when the supports of f, f' have zero intersection. Microcausality, then, depends on being able to construct such an inner product which is local in space; as above, that is position space.

But we know that the covariant representation is also local (in configuration space). Suppose now we could consider the manifold of all configuration space solutions of the complex KG equation (denote M) as a Hilbert space, with covariant inner product

$$(\varphi,\varphi')_M = i\hbar \int (\bar{\varphi}\overleftrightarrow{\frac{\partial}{\partial t}}\varphi') \, d^3x$$

(which we can not, because (.,.) is indefinite) and set up the canonical creation and annihilation operators on $\mathfrak{F}(M)$ to obtain the CCR:

$$[a(\varphi), a(\varphi')^*] = (\varphi,\varphi')_M = i\hbar \int (\bar{\varphi}\overleftrightarrow{\frac{\partial}{\partial t}}\varphi) \, d^3x$$

This too is local (in configuration space). The fields $a(\varphi)$ are covariant. What has happened to the requirement that we combine creation and annihilation operators to obtain causal fields?

Whilst the RHS of this commutator is local, we can only obtain vanishing CCR if the functions φ,φ' have supports with vanishing intersection. What prevented the fields defined as above from being causal is that we considered only positive frequency solutions, and these, as we have seen ((3.2.3)), are entire functions and do not vanish on any open set of space. The RHS can never vanish.

At the same time in order to obtain the gauge transformation properties of the fields, essential in order to define a conserved charge operator, from a gauge transformation of the Hilbert space upon which the second quantization is based, we needed to define the creation and annihilation operators as maps of the form:

$$a : (\mathcal{H}_a^+ \oplus \mathcal{H}_a^-) \longrightarrow \mathcal{L}(\mathfrak{F}(\mathcal{H}_a^+) \otimes \mathfrak{F}(\mathcal{H}_b^+))$$

(see Eq.(29)(3.3.4). This is inconsistent[1] with the canonical second quantization. But if we *could* construct creation and annihilation operators in this way, whilst preserving their fundamental defining relationships (Eq.(5) and

[1] It is a theorem (see (3.5.2) that the tensor product of $\mathfrak{F}(\mathcal{H}a)$ and $\mathfrak{F}(\mathcal{H}b)$ is isomorphic to the Fock space over the direct sum of $\mathcal{H}a$ and $\mathcal{H}b$; we could, on the basis of Section 3.3, write the creation and annihilation operators as maps from $\mathcal{H}a \oplus \mathcal{H}b$ to operators on $\mathfrak{F}(\mathcal{H}a \oplus \mathcal{H}b)$ and preserve the canonical structure. The problem is to interpret $\mathcal{H}b$ as the negative frequency solution space.

(anti)linearity properties), they would be guage covariant, covariant, *and* causal, because without the restriction to positive freqeuncy parts solutions φ can vanish on arbitrary domains at time zero.

In this section that is just what we shall do. We shall consider the creation and annihilation operators as maps:
$a: M \longrightarrow \mathcal{L}(\mathfrak{F}(\mathcal{H}_a^+)\otimes\mathfrak{F}(\mathcal{H}_b^+)$
and establish a natural correspondence between the spaces $\mathcal{H}_a^+, \mathcal{H}_b^+$ and M. Further, this construction is general, rigorous, and natural within the general framework of finding a Fock representation for the Weyl (or Clifford) algebras over a classical solution manifold. This makes it a field quantization. Because we *do* obtain a Fock representation, a transparent particle interpretation, we can see what is happening from the point of view of the 1-particle theory. Essentially, we learn that the 1-particle quantum mechanics based on the space M^- of negative frequency solutions is identical to the corresponding theory on M^+ except that *every occurence of the imaginary unit i is replaced by -i.* This does not mean the *usual* complex structure has no application in the theory. On the contrary, it is the usual complex structure which defines the gauge covariance of the fields[2].

One could proceed *ab initio* from this postulate and reconstruct (an impoverished version of) the theory which follows as a canonical second quantization of a particle-antiparticle quantum mechanics. This will give us a rather similar theory to that of Section 3.3, in which the negative frequency states have a natural rôle[3], but as mentioned above we still need the usual complex structure to give us the gauge transformation properties of the physical fields, and we would still have to motivate the construction of these on the basis of microcausality. We shall interpret

[2] For real fields, this gauge covariance and the usual complex structure are both absent.

[3] We do not here need to impose covariance, because we begin with a covariant particle-antiparticle Hilbert space.

the usual complex structure as given by the classical system that is quantized, *i.e.* c-number complex fields. This interpretation would not be possible from a pure particle approach. In (3.4.10) we shall consider the relationships between the canonical second quantized theory using the new complex structure, and that using the usual one, but we shall not formulate the theory in these terms.

We shall begin with a *brief* summary of the quantization process which we are going to use, which has already been outlined in (2.5.6) and (2.5.7). We then go on to apply this method to the real scalar and complex scalar fields ((3.4.3) and (3.4.4) respectively). For fermion fields we do not use the symplectic structure of the classical phase space (nor does the quantum theory proceed from the Weyl algebra of observables), and the procedure must be modified; this is discussed in (3.4.5). In (3.4.6) we quantize the Dirac field, and develop a new interpretation of the Dirac hole theory ((3.4.10)). The relationship to the non-relativistic theory, in particular locality, is illuminated from a different angle in (3.4.7), and in (3.4.8), which further develops the general theory of (3.4.2) and Section 2.5. From (3.4.7) on we shall be entirely concerned with interpretation.

3.4.2. Geometric quantization.

We refer to the discussion of (2.5.6) for the general relationship of complex structures (or in geometric terms, Kähler polarizations of a classical phase space), to quantization. We there showed that, given a real linear classical phase space M for a linear classical field theory, we can convert this space into a complex Hilbert space M_J if we are given a positive complex structure J, with sesquilinear inner product (Eq.(5)(2.5.7)):

$$\langle .,. \rangle_J = \omega(J.,.) - i\omega(.,.) \qquad (1)$$

where ω is the symplectic form on M. At the same time, the Weyl algebra (Eq.(3),(2.5.7)):

$$W(u)W(v) = e^{i\omega(u,v)/2\hbar} W(u+v)$$

for $u, v \in M$, in infinitesimal form becomes Eq.(4),(2.5.7)

$$[A(u),A(v)] = -i\hbar\omega(u,v) \tag{2}$$

(where $W(su) = e^{-isA(u)/\hbar}$, $s\in\mathbb{R}$, $A(u)$ e.s.a.) and we can define a concrete representation of the quantities (Eq.(6)(2.5.7)):

$$\begin{aligned} a(u) &= (2\hbar)^{-1/2}(A(u) + iA(Ju)) \\ a(u)^* &= (2\hbar)^{-1/2}(A(u) - iA(Ju)) \end{aligned} \tag{3}$$

on the Fock space over M_J as annihilation and creation operators - and thereby obtain a representation of the Segal field $A(u)$), the generator of the Weyl algebra:

$$A(u) = (\hbar/2)^{1/2}(a(u) + a^*(u)). \tag{4}$$

These equations imply the commutators (Eq.(7)(2.5.7)

$$\begin{aligned} &[a(u),a(v)] = [a^*(u),a^*(v)] = 0 \\ &[a(u),a^*(v)] = \langle u.v\rangle \end{aligned} \tag{5}$$

and a (respectively a^*) is anti-linear (linear) in its argument. Therefore the a's, a^*'s have all the properties of the canonical creation and annihilation operators defined in (1.3.4) and we may suppose they have the concrete action on $\mathfrak{F}(M_J)$ there specified. The positivity of J here means no more than that the inner product $\langle .,. \rangle_J$ is positive definite, and the multiplication of an element $u\in M$ by a complex number a+ib gives a new element of M defined by $(a+Jb)u$ (recall that J is a canonical transformation on M), so that M can be regarded as a complex linear vector space; so considered, it is denoted M_J.

As we noted previously, there is no general result which tells us that for every concrete representation of the Weyl algebra there exists a positive complex structure; nevertheless, that is the situation for the known field theories (when it is possible to define a Fock representation). A complex structure ensures that the *symplectic* transformations on M (which preserve J) can be described as *unitary* transformations on M_J and a * automorphism of the Weyl algebra (see the discussion at the end of (2.6.8)). If the dynamics is described in this way and the complex structure is invariant, the one particle evolution defines the evolution of the field as in the NRQFT.

M_J is the one-particle subspace of the Fock space; the Fock space is defined as the space of *entire* functions on M_J

(holomorphic with respect to J), and the observables are the self-adjoint operators on this space. The canonical Fock space is a representation of the space of entire functions on M_J; each term in the Taylor series expansion for an element of $\mathfrak{H}(M_J)$ defines the projection (in the canonical representation) onto an n-particle subspace, where n is the power of the term in the expansion.

This elementary formulation is rigorous when M is *finite* dimensional, in which form it was originally discovered by Fock [1928] (see Bargmann [1961] for a detailed analysis). The most simple application is to the isotropic harmonic oscillator in 1 dimension. M is then $\mathbb{R}^2$ with symplectic form $\omega = dp \wedge dq$ (with p,q, canonical coordinates), or, since we identify M with its co-tangent space,

$\omega((q,p),(q',p')) = qp'-q'p.$

The Hamiltonian is

$H = (2m)^{-1}p^2 + m\nu^2q^2$

which determines the evolution $(q,p) \longrightarrow (q_t,p_t)$, where:

$p_t = p \cos \nu t - q\, m\nu \sin \nu t$

$q_t = (p/m\nu) \sin \nu t + q \cos \nu t.$

The complex structure J is defined by

$$Jq = -p/m\nu \qquad Jp = m\nu\, q \tag{6}$$

under which M becomes a complex vector space, with $(a+ib)(q,p) = a(q,p)+bJ(q,p) = a(q,p)+b(-p/m\nu, m\nu q) = (aq-bp/m\nu, ap+bm\nu q)$. If we write

$$z = q + ip/m\nu \tag{7}$$

then $(a+ib)z = (aq-bp/m\nu+i(bq+ap/m\nu))$, which corresponds under Eq.(7) to $(aq-bp/m\nu,\ bm\nu q +ap)$; we can therefore regard M_J as the complex vector space with analytic coordinates z with the *usual* complex structure (multiplication by i; it was in this form that the representation was first discovered). In terms of the coordinates z the symplectic form becomes

$$\omega = \frac{i}{2}\, m\nu\, d\bar{z} \wedge dz.$$

M_J is a complex (finite dimensional) Hilbert space; the Hilbert space on which the observables act is the complex (infinite dimensional) vector space of entire functions on

M_J, with inner product

$$\langle \phi, \phi \rangle = \hbar^{-1} \int \overline{\phi(z)} \phi(z) e^{-m\nu z\bar{z}/2\hbar} dpdq.$$

Expanding ϕ as a Taylor series in z (contracting on upper and lower indices):

$$\phi(z^a) = \sum_{k=0}^{\infty} \left[\frac{(2\hbar)^{-k}}{k!(m\nu)^{-k}} \right]^{1/2} \eta^k_{ab\ldots c} z^a z^b \ldots z^c \tag{8}$$

where $\eta^k_{ab\ldots c}$ is symmetric with k indices, the inner product becomes:

$$\langle \phi, \phi \rangle = \Sigma_{k=0}^{\infty} \Sigma_{ab\ldots c} \bar{\eta}^k_{ab\ldots c} \eta^k_{ab\ldots c}.$$

For each vector $u \in M_J$, the creation and annihilation operators are:

$$\begin{aligned} a^*(u) &= uz \\ a(u) &= \bar{u} \frac{\partial}{\partial z} \end{aligned} \tag{9}$$

where in Eq. (9) we regard u as a complex number via Eq. (7).

The Fock-Cook representation is the infinite dimensional version of this; the action of the creation and annihilation operators is normally written in terms of the action of Eq. (9) on the power series expansion, (this is how we have defined them), but their representation in precise analogue to Eq. (9) is familiar in the specialist literature (in the so-called *Schrödinger representation* of the fields, in which the field is "diagonalized" - that is, acts by multiplication - and the canonical conjugate field acts as the generator of translations[4]).

[4] Strictly speaking, this procedure diagonalizes the creation operators; c.f. the distinction between the "real wave" and "complex wave" representations in Segal [1963], [1962]. We are using the complex wave representation, which is the one which generalizes in a natural way to the non-linear case.

3.4.3. The real scalar field.

As a classical field theory, we can regard M as either the space of all (real) solutions of the KG equation or the space of Cauchy data, that is pairs of functions on $\mathbb{R}^3$ of the form $(\varphi(\mathbf{x},0),\dot{\varphi}(\mathbf{x},0))$ (where $\dot{\varphi}(\mathbf{x},0) = \frac{\partial}{\partial t}\varphi(\mathbf{x},t)\big|_{t=0}$). We shall use *both* points of view; we use the notation (f,g) for the pair $(\varphi(\mathbf{x},0),\dot{\varphi}(\mathbf{x},0))$. The symplectic form is[5]

$$\omega(\varphi,\varphi') = -\hbar\int \varphi\overleftrightarrow{\partial_t}\varphi' d^3x = -\hbar\int (fg' - f'g)d^3x \qquad (10)$$

and the problem before us is: find a positive complex structure J so that, via Eqs.(1) - (5) above, we have a Fock space action for the Segal field, which is the generator of the Weyl algebra

$$W(\varphi)W(\varphi') = e^{i\omega(\phi,\phi')/2\hbar}\, W(\varphi+\varphi') \qquad (11)$$

To this end, consider a real solution $\varphi \in M$; one might consider taking the covariant Fourier transform:

$$\varphi(x) = \sqrt{2}(2\pi\hbar)^{-3/2}\int_{\mathbb{P}^4} e^{-ip.x/\hbar}\hat{\varphi}(p)\delta(p_0^2-m^2)d^4p$$

and using the natural complex structure $J\varphi(p) = i\varphi(p)$; however, in order for φ to be real, it is necessary that

$$\hat{\varphi}(p) = \overline{\hat{\varphi}(-p)}$$

But if φ satisfies this condition, then $i\hat{\varphi}$ does not. We cannot define J in this way. To see what must be done, we make the constraint to the mass hyperboloid $\Theta^+\cup\Theta^-$ explicit:

$$\varphi(x) = \qquad (12)$$

$$\frac{1}{\sqrt{2}}(2\pi\hbar)^{-3/2}\left[\int_{\Theta^+} e^{-ip.x/\hbar}\hat{\varphi}^+(p)d\mu^+ + \int_{\Theta^-} e^{-ip.x/\hbar}\hat{\varphi}^-(p)d\mu^-\right]$$

The reality condition requires that

$$\overline{\hat{\varphi}^+(p)} = \hat{\varphi}^-(p) \qquad (13)$$

from the above, we cannot take $J\varphi^{\pm} = i\varphi^{\pm}$, but we must rather have that:

$$J\hat{\varphi}^{\pm} = \pm i\hat{\varphi}^{\pm} \qquad (14)$$

so that Eq.(13) is preserved by the action of J. This complex structure has the same action on the configuration space solutions (by the complex linearity of Eq.(10)), so that $J\varphi^{\pm} = \pm i\varphi^{\pm}$ also. Since

$$\overline{\varphi^+}(x) = \varphi^-(x) \qquad (15)$$

and

[5] This differs by a factor i from the similar expression that we have hitherto interpreted as a sesquilinear form.

$$\varphi = \varphi^+ + \varphi^- \tag{16}$$

we can uniquely define a real solution $\varphi \in M$ by fixing a positive frequency solution $\varphi^+(x)$, or the momentum space function $\hat{\varphi}^+(p)$, in terms of which the complex structure J is the same as multiplication by i. That is, we have a correspondence between real solutions φ and elements of $\hat{M} = L^2(\Theta^+, \mu^+)$ provided by

$$\varphi(x) = \tag{17}$$
$$\frac{1}{\sqrt{2}}(2\pi\hbar)^{-3/2}\left[\int_{\Theta^+} e^{-ip.x/\hbar}\hat{\varphi}^+(p)d\mu^+ + \int_{\Theta^-} e^{-ip.x/\hbar}\overline{\hat{\varphi}^+(-p)}d\mu^-\right].$$

We can now determine the action of J on M; from Eq.(17) we see that:

$$f(\mathbf{x},0) =$$
$$\frac{1}{\sqrt{2}}(2\pi\hbar)^{-3/2}\left[\int_{\Theta^+} e^{i\mathbf{p}.\mathbf{x}/\hbar}\hat{\varphi}^+(p)d\mu^+ + \int_{\Theta^-} e^{i\mathbf{p}.\mathbf{x}/\hbar}\overline{\hat{\varphi}^+(-p)}d\mu^-\right] \tag{18}$$
$$g(\mathbf{x},o)=$$
$$-i\frac{1}{\sqrt{2}}(2\pi\hbar)^{-3/2}c/\hbar\left[\int_{\Theta^+} p_0 e^{i\mathbf{p}.\mathbf{x}/\hbar}\hat{\varphi}^+(p)d\mu^+ + \int_{\Theta^-} p_0 e^{i\mathbf{p}.\mathbf{x}/\hbar}\overline{\hat{\varphi}^+(-p)}d\mu^-\right]$$

we rewrite the second of Eqs.(18) using the operator[6]

$$R = +c(-\Delta+m^2c^2/\hbar^2)^{1/2} \tag{19}$$

(since R is positive we must use -R to obtain p_0 on the negative mass hyperboloid) to obtain:

$$g(\mathbf{x},o)=$$
$$-i\frac{1}{\sqrt{2}}(2\pi\hbar)^{-3/2}\ R\left[\int_{\Theta^+} e^{i\mathbf{p}.\mathbf{x}/\hbar}\hat{\varphi}^+(p)d\mu^+ - \int_{\Theta^-} e^{i\mathbf{p}.\mathbf{x}/\hbar}\overline{\hat{\varphi}^+(-p)}d\mu^-\right]$$

from which we conclude, under $J:\varphi^+ \longrightarrow i\varphi^+$, that

$$J:\ (f,g) \longrightarrow (-R^{-1}g, Rf) \tag{20}$$

Acting either on M or on $\hat{M}$, J obviously satisfies $J^2=-1$; further

$$\omega(J\varphi, J\varphi') = \omega(J(f,g),J(f',g')) = \omega((-R^{-1}g,Rf),(-R^{-1}g',Rf'))$$
$$= \hbar\int\left(-R^{-1}gRf' - Rf(-R^{-1}g')\right)d^3x = \hbar\int(-gf' + fg')d^3x = \omega(\varphi,\varphi')$$

where we have used the fact that R is e.s.a. on $L^2(\mathbb{R}^3, d^3x)$. Consequently, J is canonical. We may also check that

$$\omega(\varphi, J\varphi) = \omega\left[(f,g),(-R^{-1}g,Rf)\right] = \hbar\int(fRf + gR^{-1}g)d^3x$$

which is positive definite since R (and R^{-1}) is positive; therefore J is a positive complex structure and the inner product on M

[6] Recall that in our notation cp_0 has the units of energy, and x_0 has the units of length. Throughout this section p_0 can take on either sign; when we wish to indicate that p_0 is strictly positive we use the notation E, with the units of energy (i.e. $E = cp_0$ when p_0 is positive).

$$\langle\varphi,\varphi'\rangle_J = \omega(J\varphi,\varphi') - i\omega(\varphi,\varphi') = \hbar\int[(R^{-1}gg'+Rff') + i(fg'-gf')]d^3x \qquad (21)$$

is positive definite and linear (antilinear) in its second (first) entry. To check this last statement, we observe that $\langle i\varphi,\varphi'\rangle_J = \langle J\varphi,\varphi'\rangle_J = \omega(J^2\varphi,\varphi')-i\omega(J\varphi,\varphi') = -\omega(\varphi,\varphi') - i\omega(J\varphi,\varphi') = -i\langle\varphi,\varphi'\rangle_J$, whereas $\langle\varphi,i\varphi'\rangle_J = \omega(J\varphi,J\varphi') - i\omega(\varphi,J\varphi') = \omega(\varphi,\varphi') + i\omega(J\varphi,\varphi') = +i\langle\varphi,\varphi'\rangle_J$.

It is an easy calculation to show that

$$\langle\varphi,\varphi'\rangle_J = \int_{\Theta^+} \overline{\hat{\varphi}^+(p)}\hat{\varphi}'^+(p)\, d\mu^+$$

Since R is positive and Hermitian we may take its square root C, in terms of which we may also write this inner product as

$$\langle\varphi,\varphi'\rangle_J = \hbar\int \overline{\chi(x)}\chi'(x)d^3x$$

where $\chi(x) = Cf + iC^{-1}g$ (*cf.* Eq.(7)). The KG equation, which in terms of the Cauchy data reads:

$$\frac{\partial}{\partial t}f = g, \quad \frac{\partial}{\partial t}g = -R^2f \qquad (22)$$

becomes:

$$R\,\chi = i\frac{\partial}{\partial t}\chi \qquad (23)$$

that is, we obtain a Schrödinger equation with Hamiltonian $\hbar R$. Eqs.(22) are identical to Eqs.(19) of (3.2.7), that is, as the preliminary to application of the Foldy transform, whereas Eq.(23) is the Foldy transform of the positive frequency *(complex)* solutions of the KG equation. We shall consider this further below ((3.4.7)).

From the point of view of constructing a particle representation of the quantum field the essential point is that we have an appropriate complex structure with which we can define a concrete representation of the Weyl algebra and with it an action of the quantum field. Exactly as in (1.3.4) we construct the Fock space $\mathfrak{F}(M_J)$ with the creation and annihilation operators over M_J having the canonical action of Eq.(45)(1.3.4); as a Fock space representation, it is of course canonical, and the relationships defined by Eq.(3) and (5), namely:

$$A(u) = (\hbar/2)^{1/2}(a(u)+a^*(u))$$
$$[A(u),A(v)] = i\hbar\,\mathrm{Im}(\langle u,v\rangle_J) \qquad (24)$$
$$[a(u),a^*(v)] = \langle u,v\rangle_J$$

are identical to Eqs.(40),(41), and (35) of (1.3.4). We stress the fundamental feature of this approach, that the commutator of the Segal field is given by the imaginary part of the inner product, equivalently by the symplectic form on the classical solution phase space, and the complex structure gives us the real part, in the sense that
$\mathrm{Re}(\langle u, v\rangle) = \omega(Ju, v)$.

We obtain the standard representation of (3.3.2) by the isomorphism between the real solution space M and the complex Hilbert space $L^2(e^+, \mu^+)$ (that is, using Eq.(17)); the action of J on this space is just multiplication by i. As discussed at the end of (2.5.8) the symplectic transformations of M induce unitary transformations of M_J, and these induce the second quantized unitary transformations on the fields; *i.e.*
$\Gamma(U)A(u)\Gamma(U)^{-1} = A(Uu)$.
In particular, the Hamiltonian flow on M defines a 1-parameter unitary group $U_t = e^{-itd\Gamma(H)/\hbar}$ on $\mathfrak{F}(M_J)$ where $d\Gamma(H)$ is the second quantization of the Hamiltonian on M_J, that is, where H generates the unitary representation of the symplectic transformations on M corresponding to the Hamiltonian flow; H is just the quantization of the generator of this flow, which is the classical energy. We can rewrite Eq.(17) in terms of an integral (over the positive mass shell only) to make this explicit:

$$\varphi(\mathbf{x}, t) = (2\pi\hbar)^{-3/2} c \int_{\mathbb{P}^3} \left[\hat{\varphi}^+(\mathbf{p}, E/c) e^{-iEt/\hbar} + \overline{\hat{\varphi}^+(-\mathbf{p}, E/c)} e^{+iEt/\hbar} \right] e^{i\mathbf{p}\cdot\mathbf{x}/\hbar} d^3p/\sqrt{2}E$$

(in which $E = +c(\mathbf{p}^2 + m^2c^2)^{1/2}$); therefore the transformation

$$\varphi(x, 0) \longrightarrow \varphi(x, t) \qquad (25)$$

corresponding to

$$f \longrightarrow f_t = \varphi(., t)$$
$$g \longrightarrow g_t = \varphi(., t)$$

is implemented as the unitary transformation on $L^2(e^+, \mu^+)$:
$\hat{\varphi}^+(\mathbf{p}, p_0) \longrightarrow \hat{\varphi}^+(\mathbf{p}, p_0)\, e^{-iEt/\hbar}$
in which E is positive definite. The energy is positive.

What if we do not rewrite Eq.(17) in this way? In fact, let us look at the transformation properties of Eq.(12), supplemented by the constraint Eq.(13). Clearly the

transformation of Eq.(25) is given by:

$$\hat{\varphi}^{+}(p) \longrightarrow \hat{\varphi}^{+}(p)\, e^{-ip_0 ct/\hbar}$$
$$\hat{\varphi}^{-}(p) \longrightarrow \hat{\varphi}^{-}(p)\, e^{-ip_0 ct/\hbar}$$

that is, we appear to have a negative generator on the $\hat{\varphi}^{-}$ states. Since Eq.(13) holds, *i.e.* $\hat{\varphi}^{-}(p) = \overline{\hat{\varphi}^{+}(-p)}$, the second of the above is a consequence of the first, and can be considered simply a constraint[7]. In precisely the same way we can avoid use of the complex structure $J\varphi^{\pm} = \pm i\varphi^{\pm}$, and simply use multiplication by i on the φ^{+} states. But if we do look at the generator on the negative frequency parts, we must use the complex structure on these functions, and that is multiplication by -i.

One must consider more carefully the relationship between the generator of a unitary group and the complex structure of the Hilbert space. Stone's theorem in this context becomes the statement: for each one-parpameter weakly continuous unitary group U_t acting on M_J there exists an s.a. operator H acting on M_J (s.a. with respect to the inner product $\langle .,. \rangle_J$ that is), such that

$$U_t = e^{-JtH/\hbar}$$

(*cf.* the comments at the end of (2.5,8) and of (3.2.2) fn.3). Therefore on the states $\hat{\varphi}^{-}$, if $\hat{\varphi}^{-}(p) \longrightarrow \hat{\varphi}^{-}(p)\, e^{-icp_0 t/\hbar}$ with $J\hat{\varphi}^{-} = -i\hat{\varphi}^{-}$ it follows that $H = -p_0$ on $L^2(e^{-}, \mu^{-})$, that is, the Hamiltonian is after all positive, and the negative frequency states have positive energy with respect to the complex structure J.

[7] The same is true in NRQM; if $f(\mathbf{p},0) \longrightarrow f(\mathbf{p},t) = f(\mathbf{p},0)e^{-iEt/\hbar}$ then $\overline{f(\mathbf{p},0)} \longrightarrow \overline{f(\mathbf{p},t)} = \overline{f(\mathbf{p},0)}e^{iEt/\hbar}$. The following also applies to these states, the solutions to the c.c. Schrodinger equation. See also (3.4.12).

3.4.4. The complex KG field.

Consider now the complex case; M now already has a complex structure, which is multiplication by i (we shall call this the **natural complex structure**, denote J_N). From a formal point of view, we have precisely the same mass shell Fourier transform (Eq.(12) except that now the condition of Eq.(13) no longer holds. That is, we have

$$\varphi(x) = \int_{\Theta^+} e^{-ip.x}\hat{\varphi}^+(p)d\mu^+ + \int_{\Theta^-} e^{-ip.x}\hat{\varphi}^-(p)d\mu^- \qquad (26)$$

and $\hat{M} = L^2(\Theta^+,\mu^-)\cup L^2(\Theta^-,\mu^-)$; there is no connection between φ^- and φ^+. Like the real case, it is clear that as $\varphi(\mathbf{x},0) \longrightarrow \varphi(\mathbf{x},t)$ then we must have

$\hat{\varphi}^{\pm}(p) \longrightarrow \hat{\varphi}^{\pm}(p)\ e^{-ip_0 t/\hbar} = \hat{\varphi}^{\pm}(p)e^{\mp|p_0|t/\hbar}$

($\hat{\varphi}^+$ vanishes on Θ^-; $\hat{\varphi}^-$ vanishes on Θ^+; p_0 here takes positive and negative values accordingly). That is, the generator of the time evolution on $L^2(\Theta^-,d\mu^-)$ is negative, with respect to the natural complex structure, as expected.

At the same time, we also find that the inner product defined by the natural complex structure is *not* positive definite. The relevant equations for the complex case, though easy to derive, do not seem to have been written down in the literature so we state them in full. The symplectic form in covariant form is (μ=0,1,2,3):

$\omega(\varphi,\varphi') = -\frac{1}{2}\hbar\int\left(\varphi\partial_\mu\bar{\varphi}' - (\partial_\mu\varphi)\bar{\varphi}'\right)d\sigma^\mu$ + c.c.

or in terms of the (now complex) Cauchy data (f,g) on a spacelike hypersurface:

$$\omega((f,g),(f',g')) = -\frac{1}{2}\hbar\int (f\bar{g}' - g\bar{f}' + \bar{f}g' - \bar{g}f')d^3x. \qquad (27)$$

This form is constructed as the unique real antisymmetric form which is invariant under the time evolution defined by the KG equation. The natural complex structure $J_N:\varphi \longrightarrow i\varphi$, or $(f,g) \longrightarrow (if,ig)$, is canonical by inspection. The sesquilinear form defined by J_N is

$$\langle\varphi.\varphi'\rangle_N = \omega(J_N\varphi,\varphi') - i\omega(\varphi,\varphi') = \omega(i\varphi,\varphi') - i\omega(\varphi.\varphi')$$

$$= \frac{1}{2}\hbar\int\left[(if\bar{g}' - ig\bar{f}' - i\bar{f}g' + i\bar{g}f') - (if\bar{g}' + ig\bar{f}' - i\bar{f}g' + i\bar{g}f')\right]d^3x$$

$$= i\hbar\int(\bar{f}g' - \bar{g}f')d^3x \qquad (28)$$

or in covariant form[8]

$$\langle\varphi,\varphi'\rangle_N = i\int \left(\bar{\varphi}\partial_\mu\varphi' - (\partial_\mu\bar{\varphi})\varphi'\right) d\sigma^\mu.$$

Evidently $\langle\varphi,\varphi\rangle_N$ is real but it is *not* positive definite; therefore M regarded as a complex vector space is *not* pre-Hilbert. It is helpful to have an explicit expression for $\langle .,. \rangle_N$ on the covariant Fourier transform space $\hat{M}$; using Eq. (26)[9]:

$$\omega(\varphi,\varphi') = \tag{29}$$
$$-\frac{1}{2}\left[i\int_{\Theta_+}\hat{\bar{\varphi}}\hat{\varphi}'\,d\mu^+ - i\int_{\Theta_-}\hat{\bar{\varphi}}\hat{\varphi}'\,d\mu^- - i\int_{\Theta_+}\hat{\bar{\varphi}}\hat{\varphi}'\,d\mu^+ + i\int_{\Theta_-}\hat{\bar{\varphi}}\hat{\varphi}'\,d\mu^-\right]$$

$$\omega(i\varphi,\varphi') = \tag{30}$$
$$\frac{1}{2}\left[\int_{\Theta_+}\hat{\bar{\varphi}}\hat{\varphi}'\,d\mu^+ - \int_{\Theta_-}\hat{\bar{\varphi}}\hat{\varphi}'\,d\mu^- + \int_{\Theta_+}\hat{\bar{\varphi}}\hat{\varphi}'\,d\mu^+ - \int_{\Theta_-}\hat{\bar{\varphi}}\hat{\varphi}'\,d\mu^-\right]$$

we find:

$$\langle\varphi,\varphi'\rangle_N = \int_{\Theta_+}\hat{\bar{\varphi}}\hat{\varphi}'\,d\mu^+ - \int_{\Theta_-}\hat{\bar{\varphi}}\hat{\varphi}'\,d\mu^- \tag{31}$$

We now consider the complex structure $J_P:\varphi^\pm \longrightarrow \pm i\varphi^\pm$, that is $J_P:\hat{\varphi}(p)\longrightarrow i\epsilon(p_0)\hat{\varphi}(p)$ (where $\epsilon(x) = \theta(x)-\theta(-x)$). We shall call this the **particle complex structure**. In exactly the same way as in the real case, we conclude from:

$$f(\mathbf{x},0) = \tag{32}$$
$$\frac{1}{\sqrt{2}}(2\pi\hbar)^{-3/2}\left[\int_{\Theta_+} e^{i\mathbf{p}.\mathbf{x}/\hbar}\hat{\varphi}(p)d\mu^+ + \int_{\Theta_-} e^{i\mathbf{p}.\mathbf{x}/\hbar}\hat{\varphi}(p)d\mu^-\right]$$

$$g(\mathbf{x},o)=$$
$$-\frac{i}{\sqrt{2}}(2\pi\hbar)^{-3/2}c/\hbar\left[\int_{\Theta_+} p_0 e^{i\mathbf{p}.\mathbf{x}/\hbar}\hat{\varphi}(p)d\mu^+ + \int_{\Theta_-} p_0 e^{i\mathbf{p}.\mathbf{x}/\hbar}\hat{\varphi}(p)d\mu^-\right]$$

(*cf.* Eq. (18)) and introducing the operator R of Eq. (19), to obtain:

$$g(\mathbf{x},o)=$$
$$-\frac{i}{\sqrt{2}}(2\pi\hbar)^{-3/2}\,R\left[\int_{\Theta+} e^{i\mathbf{p}.\mathbf{x}/\hbar}\hat{\varphi}(p)d\mu^+ - \int_{\Theta-} e^{i\mathbf{p}.\mathbf{x}/\hbar}\hat{\varphi}(p)d\mu^-\right],$$

[8] The differences between $\langle .,. \rangle_N$ and $\omega(.,.)$ should be carefully noted; ω is real, and not sesquilinear. Further, $\omega = -\mathrm{Im}(\langle .,. \rangle_N)$ as required. Of course this form, either on M or on $\hat{M}$, is what was taken as the "inner product" for the scalar theory since its inception.

[9] Here and in what follows we suppress the superscripts $\pm$ on the ϕ's and their covariant transforms, the φ's; the measure of the integral will indicated which sign of the frequency they refer to. When they occur outside of any integral, they will be general solutions with both positive and negative frequency parts.

that the action of J_ρ on the Cauchy data is, as before,

$J_\rho : (f,g) \longrightarrow (-R^{-1}g, Rf)$.

We can evaluate the new sesquilinear form $\langle .,. \rangle_\rho$ using this action; from Eq.(27) we see that the real part of $\langle .,. \rangle_\rho$ is:

$$\omega(J_\rho(f,g),\ (f',g')) = -\frac{1}{2}\hbar \int [(-R^{-1}g\bar{g}' - Rf\bar{f} - \overline{R^{-1}g}\ g' - \overline{Rf}\ f']d^3x \tag{33}$$

Clearly $\omega(J_\rho\varphi,\varphi)$ is positive definite; M_ρ is therefore pre-Hilbert. On $\hat{M}$ we find from Eq.(29) that:

$$\omega(J_\rho\varphi,\varphi') = \frac{1}{2}\hbar\left[\int_{\Theta^+}\hat{\bar{\varphi}}\hat{\varphi}'\,d\mu^+ + \int_{\Theta^-}\hat{\bar{\varphi}}\hat{\varphi}'\,d\mu^- + \int_{\Theta^+}\hat{\bar{\varphi}}\hat{\varphi}'\,d\mu^+ + \int_{\Theta^-}\hat{\bar{\varphi}}\hat{\varphi}'\,d\mu^-\right]$$

(which is obviously positive definite when $\varphi=\varphi'$) so that

$$\langle \varphi,\varphi' \rangle_\rho = \int_{\Theta^+}\hat{\bar{\varphi}}\hat{\varphi}'\,d\mu^+ + \int_{\Theta^-}\hat{\bar{\varphi}}\hat{\varphi}'\,d\mu^- \tag{34}$$

again, the differences between this and Eq.(31) should be carefully noted; the negative mass hyperboloid not only contribute with opposite sign (which is what makes $\langle .,. \rangle_\rho$ positive definite), but also we have $\hat{\bar{\varphi}}\hat{\varphi}'$ in the integrand and not $\hat{\varphi}\hat{\bar{\varphi}}'$; this fact is crucial. Despite the appearance to the contrary in Eq.(34), $\langle .,. \rangle_\rho$ is still linear in its second entry.

The positivity of the generator of time evolutions follows as indicated in the closing remarks of (3.4.3). From Stone's theorem we conclude that the Schrödinger equation is:

$$H\varphi = J_\rho \hbar \frac{\partial}{\partial t}\varphi \tag{35}$$

so that the Hamiltonian is given by

$H\hat{\varphi}(p) = \epsilon(p_0)cp_0\hat{\varphi}(p)$

on $\hat{M}$; it is clearly a positive operator. In view of this and bearing in mind that the completion of M_ρ defines the one-particle subspace of the Fock space, we make the fundamental assumption: **the restriction of $\hat{\varphi}$ to Θ^+ (respectively Θ^-) defines a particle (antiparticle) state.**

At this point let us make clear the general situation for observables X which are defined as generators for groups of 1-parameter transformations on M; these transformations are symplectic, and become unitary groups on M_J (with respect to the complex structure on M); the e.s.a. operators $\mathcal{X}$ which generate these groups are then the quantum analogues of the generators X. The symplectic transformations are of course

independent of J; therefore in a sense the associated operators are also independent of J, (namely, that they can be specified in an unambiguous way without reference to any explicit complex structure), but their properties, *qua* generators of *unitary* transformations on M_J, depends on J (in particular *positivity*).

In the present context we have two complex structures; the natural and the particle ones. For these the relationships between the quantum observables is particularily simple: if $\mathfrak{X}^N$ is associated with the generator of the symplectic transformations S_g on M, self adjoint and generating the unitary transformation U_g^N on M_N with respect to its natural complex structure J_N, then

$$-J_N J_P \mathfrak{X}^N = \mathfrak{X}^P \qquad (36)$$

is the self-adjoint generator of the unitary transformations U_g^P on M_P, likewise associated with the symplectic tranformations S_g on M[10]. In a *set-theoretic* (pointwise) sense the two ($\mathfrak{X}^P$ and $\mathfrak{X}^N$ related through Eq.(36)) generate the *same* evolution on M_P and M_N respectively; each of these spaces is at the same time the space M, and the pointwise evolution on each is just the symplectic evolution on M.

We now introduce the creation and annihilation operators with respect to the *particle* complex structure. The Segal field $A(\varphi)$, which is the generator of the Weyl algebra, has the canonical relationship to these operators, defined by Eqs.(2) and (3). From an abstract point of view one defines the creation and annihilation operators via Eq.(3). But since we have a canonical representation of these operators on $\mathfrak{F}(M_P)$ we work the other way, and define $A(\varphi)$ as:

$$A(\varphi) = (\hbar/2)^{1/2}(a(\varphi)+a^*(\varphi))$$

where φ has in general both positive and negative frequency parts. If we make this decomposition explicit, that is, we write $\varphi = \varphi^+ + \varphi^-$, we obtain:

$$A(\varphi) = (\hbar/2)^{1/2}\left[a(\varphi^+)+a(\varphi^-) + a^*(\varphi^+)+a^*(\varphi^-)\right]. \qquad (37)$$

[10] Eq.(36) is generally true for any two complex structures J_1, J_2 which commute.

Its physical interpretation, as a sum of creation and annihilation operators for particle and anti-particle states, is clear; the Segal field $A(\varphi)$ is of course real linear only; it is a map on the *classical* phase space.

To obtain the *physical* quantum fields consider the creation and annihilation fields defined with respect to the *natural* complex structure. Our first definition is in terms of the field A, since M_N is not pre-Hilbert, that is we use Eq.(2):

$$b(\varphi) = (2\hbar)^{-1/2}(A(\varphi)+iA(J_N\varphi))$$
$$b^*(\varphi) = (2\hbar)^{-1/2}(A(\varphi)-iA(J_N\varphi)) \tag{38}$$

We can now obtain a concrete action for these operators, by writing out $A(\varphi)$ and $A(J_N\varphi)$ in terms of the a's and a^*'s defined with respect to the particle complex structure (that is, we express the b's and b^*'s as operators on $\mathfrak{F}(M_P)$). Since we obtain expressions like $a(i\varphi)$, and a, (respectively a^*) are anti-linear (linear) with respect to the particle complex structure J_P, we must write such expressions in terms of J_P. We write J_P as:

$$J_P = iP^+ - iP^- \tag{39}$$

where $P^\pm$ are projection operators onto the positive and negative frequency parts. In this way we obtain:

$$b(\varphi) = \tfrac{1}{2}\left[a(\varphi) + a^*(\varphi) + ia(J_P(P^+-P^-)\varphi) + ia^*(J_P(P^+-P^-)\varphi)\right]$$
$$= \tfrac{1}{2}\left[a(\varphi) + a^*(\varphi) + a((P^+-P^-)\varphi) - a^*((P^+-P^-)\varphi)\right]$$
$$= a(\varphi^+)+a^*(\varphi^-) \tag{40}$$

and similarily

$$b^*(\varphi) = a(\varphi^-)+a^*(\varphi^+).$$

These are the physical scalar fields of (3.3.2); their action, as the linear sum of an annihilation operator for particles, and a creation operator for antiparticles, now has a clear origin in their definition in terms of the natural complex structure. The gauge transformations on the fields is a consequence of the gauge transformation on M equipped with its *natural* complex structure; that is, under $\varphi \longrightarrow e^{J_N\theta}\varphi$ we have

$$b(e^{J_N\theta}\varphi) = a(e^{i\theta}\varphi^+) + a^*(e^{i\theta}\varphi^-) = a(e^{J_P\theta}\varphi^+) + a^*(e^{-J_P\theta}\varphi^-)$$
$$= e^{-i\theta}a(\varphi^+)+e^{-i\theta}a^*(\varphi^-) = e^{-i\theta}b(\varphi) \tag{41}$$

whereas:

$$b^*(e^{J_N\theta}\varphi) = a^*(e^{i\theta}\varphi^+) + a(e^{i\theta}\varphi^-) = a^*(e^{J_P\theta}\varphi^+) + a(e^{-J_P\theta}\varphi^-)$$

$$= e^{i\theta}b(\varphi)$$

Gauge transformations with respect to the particle complex structure, $\varphi \longrightarrow e^{J_P\theta}\varphi$, on the other hand, define the canonical transformations $a(\varphi) \longrightarrow e^{-i\theta}a(\varphi)$, $a^*(\varphi) \longrightarrow e^{i\theta}a^*(\varphi)$. We shall consider the gauge theory in more detail shortly ((3.4.9)).

3.4.5. Fermion systems.

There are profound differences between fermion and boson fields, and we shall barely scratch the surface in what follows. In brief, we shall develop a quantization which minimalises these differences.

The basic idea is to develop a Fock-Cook particle representation of an infinite dimensional Clifford algebra, that is, a Clifford algebra over an infinite dimensional real vector space. This abstract object we suppose is the fermion analogue of the Weyl algebra. We shall not attempt to interpret the Fock space in terms of a power series expansion of holomorphic functions on this real vector space, equipped with a suitable complex structure, but as is clear from the foregoing we can obtain a particle representation without the explicit use of this machinary.

The immediate difference is that whereas starting from a symplectic phase space one is given a symplectic form, the imaginary part of the inner product on the complexified phase space, a Clifford algebra presents naturally a different object, namely a *symmetric* non-degenerate bilinear form. The natural invariance group of the algebra is then the *orthogonal* transformations, rather than the symplectic group. The representation theory is in some respects much simpler than for Weyl algebras: since the Clifford algebra is simple, every non-zero representation is faithful, so that every non-zero representation is weakly equivalent. However, as discussed in (2.3.10), weakly equivalent representations may yet differ in their physical

interpretation, and as von Neumann algebras may be of different type (see Segal [1963], Shale and Stinespring [1964], and the material of Section 3.5).

Following Bongaarts [1972], we suppose we have two things: a real linear vector space M, and a nondegenerate symmetric bilinear form S on M. When M is finite dimensional, the Jordan-Wigner uniqueness theorem applies and there is only one irreducible representation for each pair M,S (when $M = \mathbb{R}^4$ and S = g we obtain the representation defined by the algebra of the γ matrices). When M is infinite dimensional, we proceed as follows: we suppose there is an associative algebra $\mathcal{A}$ with the properties:

(i) $\mathcal{A}$ has an identity $\mathbb{1}$.

(ii) there exists a real linear injection A: $M \longrightarrow \mathcal{A}$ such that $\mathcal{A}$ is generated by the elements $A(u)$, where $u \in M$.

(iii) $[A(u),A(v)]_+ = S(u,v)$

(*cf.* Eq.(53) (1.3.4). This algebra can be further equipped with an adjunction with respect to which the A's are self-adjoint. We define a set of states $\mathfrak{S}$ as positive linear functionals on this algebra and in the manner of (2.3.5) thereby introduce a norm on $\mathcal{A}$. Completion of $\mathcal{A}$ then defines a C^*-algebra (which we also denote $\mathcal{A}$). An orthogonal transformation on M (*i.e.* a linear transformation which leaves S invariant) will generate a * automorphism of $\mathcal{A}$ as in (2.5.2), the second quantization of the associated one-particle orthogonal transformation. The existence of a positive compatible complex structure J on M (where compatible now means that $S(Ju,Jv) = S(u,v)$) will ensure which preserve J that the orthogonal transformations of M become unitary transformations of M_J.

Whereas in the Weyl case the symplectic form defines the imaginary part of the inner product, and the complex structure the real part, we now have that S defines the real part and the complex structure the imaginary part. In particular S(J.,.) is anti-symmetric and

$$<.,.>_J = S(.,.)+iS(J.,.) \qquad (42)$$

is sesquilinear (linear in its second entry). An important difference from the boson theory is that M_J is pre-Hilbert

if and only if S is positive definite; the complex structure has no rôle to play here. However, the complex structure *does* determine the positivity (or otherwise) of the generators of the unitary representations (in M_J) of orthogonal transformations (on M).

3.4.6. The Dirac field.

We take for M the space of Cauchy data for the Dirac equation, with Cauchy data given by the time-zero solutions. We consider M as a *real* vector space, and for S the real linear, bilinear, non-degenerate form:

$$S(\psi,\psi') = \mathrm{Re}\left[\int \Sigma_i \overline{\psi_i(\mathbf{x})}\psi_i(\mathbf{x})\, d^3x\right] \qquad (43)$$

(the sum is over the 4 spinor components of ψ); the procedure is now identical to the boson case, except that in consideration of the natural complex structure on M, since S is positive definite, M_N is pre-Hilbert. Of course, with respect to the natural complex structure, we know that the evolution $\psi(\mathbf{x}) \longrightarrow \psi(\mathbf{x},t)$ (which is orthogonal with respect to S) becomes the unitary evolution:

$$\psi(\mathbf{x}) \longrightarrow e^{-iHt/\hbar}\psi(\mathbf{x})$$

where $H = (-ic\hbar\gamma^k\frac{\partial}{\partial x^k} + mc^2)\gamma^0$, $k = 1,2,3$, noting that by Eq.(42) the inner product $\langle .,. \rangle_N$ on M induced by the natural complex structure is given by:

$$\langle \psi,\psi' \rangle_N = \int \Sigma_i \bar{\psi}_i\psi_i d^3x \ . \qquad (44)$$

H is of course not positive, so the natural complex structure is not the correct one. We take instead the complex structure of Eq.(39) and obtain the new inner product:

$$\langle \psi,\psi' \rangle_P = S(\psi,\psi')+iS(J\psi,\psi') = \mathrm{Re}(\langle \psi,\psi' \rangle_N)+i\,\mathrm{Im}(\langle P^+-P^-\psi,\psi' \rangle_N). \qquad (45)$$

When ψ,ψ' are the Cauchy data for a positive frequency solution of the Dirac equation $\langle \psi,\psi' \rangle_P = \langle \psi,\psi' \rangle_N$; for a negative frequency solution rather:

$$\langle \psi,\psi' \rangle_P = \overline{\langle \psi,\psi' \rangle}_N = \langle \psi',\psi \rangle_N \qquad (46)$$

(*cf.* Eq.(34)). It is now clear that the negative frequency solutions have positive energy, since the evolution $\psi \longrightarrow e^{-iHt/\hbar}\psi$, which in terms of the particle complex structure is (*cf.* Eq.(36)):

$$\psi \longrightarrow e^{-J_P(-J_N J_P H)t/\hbar}$$

has generator $-J_N J_P H = P^+H - P^-H$ (that is, the sign of H acting on negative frequency solutions is reversed). Explicitly, with $\psi^\pm = P^\pm\psi$, we have:

$$\langle\psi, -J_N J_P H\psi\rangle_P = \langle\psi^+, H\psi^+\rangle_N - \langle\psi^-, H\psi^-\rangle_N \qquad (47)$$

The quantum fields are defined identically to the boson case, except of course the creation and annihilation operators are defined on the skew-symmetric algebra over M_P, and in particular one has the precise analogue of Eqs.(37), (38), (40) (with the factor $\hbar$ removed). The Segal field A (generating the Clifford algebra) satisfies:

$$[A(\psi), A(\psi')]_+ = S(\psi.\psi').$$

The annihilation operator defined with respect to the *natural* complex structure

$$b(\psi) = \frac{1}{\sqrt{2}}\left(A(\psi)+iA(J_N\psi)\right)$$

is the physical quantum field; it may be written in terms of the creation and annihilation operators defined with respect to the *particle* complex structure exactly as in the Boson case:

$$b(\psi) = a(\psi^+)+a(\psi^-).$$

The gauge transformation properties are also exactly the same as for the scalar field.

Henceforward the creation and annihilation operators, and also the sesquilinear form, defined with respect to the particle (respectively natural) complex structure will be called particle (natural)[11]. We shall at times speak of "P-gauge invariance" and "N-gauge invariance", to indicate the complex structure with respect to which the gauge transformations are defined.

[11] From the foregoing it is clear that the natural creation and annihilation operators are actually the physical fields, in the terminology of Section 3.3. However the terminology "physical creation and annihilation operators" is confusing: the phrase suggests those creation and annihilation operators actually used in physical applications, which are the particle ones.

3.4.7. Causality and locality; formal considerations.

So far we have made no mention of causality, which played such an important rôle in the construction of the physical fields in Section 3.3. The situation is essentially as follows: the symplectic form (boson case) and symmetric bilinear form (fermion case) are local, in the sense that $\omega(u,v)$ and $S(u,v)$ vanish when the supports of the Cauchy data for u and v have zero intersection. We shall now examine the various (anti)commutators given above, and they shall be called causal (or otherwise) depending on whether or not they vanish when the supports of the Cauchy data for the fields have zero intersection. This is not entirely satisfactory as a definition of causality, since all these (anti)commutators are for equal-times; one would like to see locality expressed as a condition on the (anti)commutativity of the local algebras associated with arbitrary spacetime domains (*cf.* (2.5.3), or in terms of the point fields at arbitrary points in spacetime. As indicated in (2.5.7) it is not too difficult to define a correspondence between the present theory and the theory of point fields; we shall not, however, go into the details of the correspondence, but refer to the qualitative discussion of (2.5.6) and (3.5.2), and Segal [1967a p.142-7], Woodhouse [1980 7.10]. The criterion we have given is in agreement with a more careful definition for equal times, and because of the covariance of the theory our conclusions will hold at arbitrary times. This will no longer be true when we consider the fields on position space, however (see below).

From this criterion it is obvious that (anti)commutators which involve finite polynomials in the Segal fields and their deriviatives are causal. However the indefinite sesquilinear form defined by the natural complex structure for the complex scalar field, given in Eq.(28):

$$\langle\varphi,\varphi'\rangle_N = \langle(f,g),(f',g')\rangle_N = i\hbar\int(\bar{f}g' - \bar{g}f')d^3x$$

is also local (the symplectic form is, we recall, minus the imaginary part of $\langle .,. \rangle$). Therefore the natural creation and annihilation operators are also causal (Eq.(5)):

$[a(\varphi),a^{*}(\varphi')] = \langle\varphi.\varphi'\rangle_N$

Similar comments apply to the Dirac field. On the other hand, in the inner product defined by the particle complex structure (Eqs.(1),(27),(33)):

$\langle\varphi.\varphi'\rangle_P = \omega((f,g),J(f',g')) + i\omega((f,g),(f',g')) =$

$= \int(f\overline{Rf'} + g\overline{R^{-1}g'} + \bar{f}Rf' + \bar{g}R^{-1}g')d^3x + i\int(f\bar{g}' - g\bar{f} + \bar{f}g' - \bar{g}f')d^3x$

we see the intervention of the operator R defined by Eq.(19):

$R = + c(-\Delta+m^2c^2/\hbar^2)^{1/2}$.

This operator has been subjected to a systematic analysis by Segal and Goodman [1965], who were able to show that R is not only non-local, but **anti-local**, in the sense that for non-zero f in the domain of R, $(\text{supp.}f)^c \cap (\text{supp.}Rf)^c = \emptyset$ (here c indicates the set theoretic complement); that is, f and Rf cannot simultaneously vanish in any region unless f is zero . They were in fact able to prove the following theorem:

Theorem 3.4.1.

The operator $(-\Delta + m^2)^{\lambda}$ is anti-local in $L^2(\mathbb{R}^n,d^nx)$ when n is odd and λ is a non-integral real number.

It follows that for any $S \in (\text{supp.}f)^c$ that $(Rf)(x) \neq 0$ for all $x \in S$, and that as a consequence the *real* part of $\langle\varphi,\varphi'\rangle_J$ is non-local; hence the particle creation and annihilation operators are *not* causal.

We did not make explicit use of the operator R in the case of the Dirac field. However this operator occurs in the configuration space version of the projection operators $P^{\pm}$ used to define the particle complex structure, which intervenes in the particle inner product (Eq.(45)):

$\langle\psi,\psi'\rangle_P = S(\psi,\psi') + iS(J\psi,\psi') = \text{Re}(\langle\psi,\psi'\rangle) + i\text{Im}(\langle P^{+}-P^{-}\psi,\psi'\rangle)$.

We conclude that in the Dirac case also, the particle creation and annihilation operators are not causal.

These conclusions agree with the conclusions of Section 3.3, where we found that only the appropriate combinations of creation and annihilation operators obeyed causal

commutation relationships; these "appropriate combinations" are of course precisely the natural creation and annihilation operators, the physical fields.

We now turn to another aspect of the concept of locality, which was intimated in (3.4.3) above (Eqs.(22),(23)) in connection with the real scalar field and the Foldy transform of Section 3.2.

We first explore the connection with the Foldy transform in a little more detail. As we have seen, for the *real* scalar field, given Cauchy data (f,g) we define the function

$\chi(x) = Cf + iC^{-1}g$

where $C^2=R$, in terms of which the inner product (Eq.(21)):

$<\varphi,\varphi'>_P = \hbar \int [(fRf'+gR^{-1}g') + i(fg'-gf')]d^3x$

becomes

$$<\varphi,\varphi'>_P = \hbar \int \overline{\chi(q)}\chi(q)\ d^3x. \tag{48}$$

Using the KG equation in the form of Eq.(22) we find:

$i\frac{\partial}{\partial t}\chi = iC\frac{\partial}{\partial t}f - C^{-1}\frac{\partial}{\partial t}g = iCg + C^{-1}Rf = R\chi$

so that using the Cauchy data $\chi(q)$ rather than $(f(x),g(x))$ we have the canonical Schrödinger equation[12] expressing the evolution on the space $L^2(\mathbb{R}^3,d^3q)$. We also note that the *particle* inner product (Eq.(48) has become local in this representation (which is position space, in the terminology of Section 3.2). This inner product is therefore *q-local*. On the other hand, $<.,.>_N$ is non q-local.

Let us try to do the same thing for the *complex* scalar field. We now have the particle inner product:

$$<\varphi,\varphi'>_P = \tag{49}$$

$$\hbar\int(f\overline{Rf}' + g\overline{R^{-1}g}' + \bar{f}Rf' + \bar{g}R^{-1}g')d^3x + i\hbar\int(f\bar{g}' - g\bar{f}' + \bar{f}g' - \bar{g}f')d^3x\ .$$

We define:

$$\chi = Cf + iC^{-1}g \tag{50}$$

$$\zeta = Cf - iC^{-1}g \tag{51}$$

in terms of which we find:

$$<\varphi,\varphi'>_P = \hbar\int \overline{\chi(q)}\ \chi(q)'d^3q + \hbar\int \zeta(q)'\ \overline{\zeta(q)}d^3q \tag{52}$$

[12] By inspection $\hbar R$ is the Hamiltonian H; multiplying through by $\hbar$ we obtain the Schrodinger equation in the proper units.

(*cf.* Eq.(34)). The KG equation

$$\frac{\partial}{\partial t}f = g, \quad \frac{\partial}{\partial t}g = -R^2 f$$

becomes:

$$i\frac{\partial}{\partial t}\chi = R\chi \qquad (53)$$
$$i\frac{\partial}{\partial t}\zeta = -R\zeta$$

which establishes that ζ is the negative frequency part of the solution φ with Cauchy data (f,g). Equivalently, we may write:

$$J_P \frac{\partial}{\partial t}\begin{pmatrix}\chi\\ \zeta\end{pmatrix} = R\begin{pmatrix}\chi\\ \zeta\end{pmatrix}. \qquad (54)$$

Despite the si-milarities to the Foldy transformed KG equation, in the later t formalism the sesquilinear form is indefinite. Of course, in the the treatment of Section 3.2 this form is the *natural* one, *i.e.*

$$\langle\varphi,\varphi'\rangle_N = i\hbar\int(\bar{f}g'-\bar{g}f')d^3x.$$

In terms of the pair χ,ζ it may be written:

$$\langle\varphi,\varphi'\rangle_N = \hbar\int \overline{\chi(\mathbf{q})}\chi(\mathbf{q})'d^3q - \hbar\int \overline{\zeta(\mathbf{q})}\,\zeta(\mathbf{q})'d^3q. \qquad (55)$$

The relationship between Eq.(53) and Eq.(55) is precisely the same as that between Eq.(34) and Eq.(31); at the same time we have recovered the Foldy transformed scalar theory.

Consider again the meaning of the functions $\chi = Cf + iC^{-1}g$; for simplicity we confine our attention to the real scalar field. To begin with, we note that the action of the particle complex structure $J_P:(f,g) \longrightarrow (-R^{-1}g, Rf)$ does *not* preserve the support of the Cauchy data; in other words, if we suppose φ is localized in the region $S \subseteq \mathbb{R}^3$, just in case its Cauchy data has support in S, then $J_P\varphi$ is not localised in S. On the other hand, if we suppose χ is localized in the region $Q \subseteq \mathbb{R}^3$, just in case its Cauchy data $(\ell,q) = (Cf, C^{-1}g)$ has support in Q, then under the action of J_P we find that:

$$J_P:(\ell,q) \longrightarrow (-CR^{-1}g, C^{-1}Rf) = (-q,\ell) \qquad (56)$$

so that $\text{supp.}J_P(\ell,q) = \text{supp.}(\ell,q)$; the corresponding action on χ is:

$$J_P:\chi \longrightarrow i\chi.$$

In other words, the transformation on the space of Cauchy data converts the non-local action of the particle complex

structure into a local form[13]. According to the precepts of the particle theory, the solutions of the KG equation must be associated with a complex vector space such that the localization properties of the system (as all other physically meaningful quantities) are independent of the phase, the first definition of locality in terms of the support of the Cauchy data (f,g) is categorically unacceptable. This notion of locality is c-locality; locality as defined with respect to the supports of the data $(\mathfrak{f},\mathfrak{g})$ is q-locality[14].

Our definition of causality can easily be modified to q-local integrands, but it will no longer generalize to arbitrary spacetime domains (*i.e.* even if the equal-time (anti)commutators vanish, it will not follow that fields q-local to causally disjoint domains at different times will vanish). We shall therefore speak of *q-causality*; because of the limitation of this notion to a given spacelike hypersurface, it is essentially the notion of causality applicable to NRQFT (a better term might be *Galillean* causality). For the real scalar field, which is of course the same thing as the Segal field, we find that the commutators of the creation and annihilation operators become causal with respect to position space, but they are no longer covariant. The corresponding real field (the Segal field generating the Weyl algebra over the classical fields on position space) is also q-causal. This is in accordance with our results of Section 3.2. Similar conclusions apply to the complex scalar and Dirac fields.

Because of the lack of covariance, q-causality may appear of little consequence, other than in the context of the relationship of RQFT to the non-relativistic theory. There

[13] One might think that in this way one converts the particle complex structure into the natural one; this would be a mistake. There is no natural complex structure for the real scalar field, and for the complex scalar field $J_p:\zeta \longrightarrow -i\zeta$.

[14] it was in this context that Segal proposed this terminology.

is, however, a further consideration, which strengthens the requirement of commutativity, which came to light following the investigations of Borchers, Haag and Schroer, Reeh and Schlieder, and Araki, in the early 60's. To introduce this idea it is helpful to use the notion of the generating functional for the representation theory of the Weyl algebra. We shall need this anyway to further clarify the concept of a particle representation.

3.4.8. Causality and locality; representation theory.

We pick up some of the threads of Section 2.5. Viewed abstractly, a Weyl system is a pair $\mathcal{W}, \mathfrak{S}_{reg}$, where $\mathfrak{S}_{reg}$ are the regular states on $\mathcal{W}$; we recall (2.5.8) that the regularity property can be defined abstractly. From the general representation theory of C^*-algebras, we know that any representation can be obtained via the GNS construction (2.3.8) for some state $f \in \mathfrak{S}$, or for a weakly continuous representation of $\mathcal{W}$ for a state $f \in \mathfrak{S}_{reg}$. For a Weyl system over M, the generating functional ρ was defined (2.5.8) as the functional:

$\rho(u) = \langle f; W(u) \rangle$, $f \in \mathfrak{S}_{reg}$

where $\langle f; . \rangle$ is the abstractly defined action of an element $f \in \mathfrak{S}$ on $\mathcal{W}$. Any regular state defines a generating functional, and, as we saw in (2.5.8), the generating functionals can be defined abstractly, such that any generating functional determines a regular state on $\mathcal{W}$. When f is a vacuum state, that is when we have a particle representation for the Weyl system, then ρ directly defines the vacuum expectation values of polynomials in the fields.

Within the present framework, we may characterize the vacuum as that state which is invariant under *every* symplectic transformation on M (therefore, that state which is invariant under every unitary transformation of M_J). As a functional on M_J, therefore, it can only have the action $u \longrightarrow \langle u, u \rangle_J$. For the Fock representation of $\mathcal{W}$ on $\mathfrak{F}(\mathcal{H})$ indeed it is of the form (Emch [1972]):

$\rho(f) = e^{-\langle f, f \rangle^2/4}$, $f \in \mathcal{H}$

and in the present theory, when $\mathcal{H} = M_J$, we obtain

$$\rho(u) = e^{-\langle u,u\rangle_J^2/4} = e^{-\omega(Ju,u)^2/4}. \tag{57}$$

As remarked in (2.5.9), we see that the complex structure plays a critical rôle in defining the vacuum.

To return to the notions of locality discussed above, we also see that when $J = J_P$, $\rho(u)$ is *not* c-local. Yet the the vacuum expectation values are given by Eq.(5)(2.5.8):

$$(\hbar/i)^n\left\{\frac{\partial^n}{\partial t_1\ldots\partial t_n}\sum_{i,j}\rho(t_1u_1+\ldots+t_nu_n)\,e^{(it_it_j\omega(u_i,u_j)/2\hbar)}\right\}_{t_1=\ldots=t_n}$$
$$= (\varnothing, A(u_1)\ldots A(u_n)\varnothing)_P \tag{58}$$

where $\varnothing$ is the vacuum and $(.,.)_P$ is the inner product of $\mathfrak{H}(M_P)$. Therefore the non c-locality of ρ in u actually means that when the u's are c-local to causally disjoint regions, nevertheless (and despite the commutativity of the fields A) the vacuum expectation values of products of the $A(u)$'s reflect a coupling of the $A(u)$'s.

This fact is well known; it is the basis of the well-known feature of vacuum expectation values, that they *cannot* be written in the form[15]:

$$\langle\varnothing, A(u_1)\ldots A(u_n)\varnothing\rangle = \langle\varnothing, A(u_1)\varnothing\rangle\ldots\langle\varnothing, A(u_n)\varnothing\rangle \tag{59}$$

However if the fields are q-localized to disjoint regions, that is $\text{supp.}(f_i,g_i)\cap\text{supp.}(f_j,g_j) = \emptyset$ for $i,j = 1,\ldots,n$, then since

$$\langle(f_i,g_i),(f_j,g_j)\rangle_P = \hbar\int[f_if_j + g_ig_j + i(f_ig_j - g_if_j)]d^3q$$

and the integrand is local, all cross-terms vanish in this case and the generating function factorizes:

$$\rho(t_1u_1+\ldots t_nu_n) = \rho(t_1u_1)\ldots\rho(t_nu_n) \tag{60}$$

in which case the decomposition of Eq.(59) is indeed possible.

The work of Borchers *et al* mentioned above culminated in a theorem due to Reeh and Schlieder [1962]. This may be

[15] although this may be possible in an asymptotic limit; this is the content of the famous cluster decomposition property, c.f. Ruelle [1962], Araki, Hepp and Ruelle [1962]).

formulated in a general way as follows. Their idea was to define the set of states which are localized in a finite volume V during a time interval τ as the subspace $\mathfrak{h}_{V,\tau}$ obtained by applying the local algebra $\mathcal{A}(B)$ (see (2.5.3)) where B is the spacetime region $V\times\tau$ to the vacuum $\varnothing$. To their surprise the authors found that $\mathfrak{h}_{V,\tau}$ is not a proper subspace but the whole Hilbert space $\mathfrak{F}(\mathcal{H})$; in other words one can generate any state whatsoever (c-localized in any region of spacetime) from field operators c-localized to an arbitrarily small region in spacetime. Subsequently other authors attempted to circumvent this difficulty by applying not the whole algebra $\mathcal{A}(B)$ but a proper subset; for example the unitaries in $\mathcal{A}(B)$ (Knight [1961]) or a subset which, roughly speaking, generates a compact set of states applied to the vacuum (Haag and Swieca [1965]).

The present theory of q-localisation does not resolve this apparent paradox. However, Segal and Goodman [1965] were able to show that, when $\tau \longrightarrow 0$, the algebra $\mathcal{A}^q(B)$ of fields q-localized to B applied to the vacuum indeed generates a proper subset of states, and one which is moreover q-localized to B. The fact that this is only true when $\tau = 0$ reflects the fact that, as discussed in Section 3.2, the Newton-Wigner localized states can propagate outside the light cone, which may be seen as a consequence of their non-covariant character.

At the same time there is another result, initiated by studies of Haag and Schroer [1962], which showed that the von Neumann algebra generated by the local algebra $\mathcal{A}(B)$ cannot be of type I, and therefore must be of type II_∞ or III_∞, in contrast to the situation for the q-local algebra $A^q(B)$ (which is affiliated to a ring of type I_∞; see the decomposition theory of (2.4.7) for details of the von Neumann classification theory).

The concept of localization remains, in the view taken here, as *the* philosophically problematic feature of linear field theory; clearly q-localization has many desirable

properties, but because of its non-covariant features, this is *not* the notion of locality used in the definition of microcausality, which is c-locality. The physics community remains curiously indifferent to this fundamental difficulty in what is supposed to be a physically trivial theory (the linear field).

The fact that *all* of the difficulties of locality are resolved if one confines attention to the local properties of fields and particles on a fixed spacelike hypersurface, is presumably what makes the familiar concepts of NRQM approximately applicable to the world of (low-energy) experience. In this sense the partial resolution which we have achieved is no more than that absolutely necessary to render consistent the application of the non-relativistic theory to such phenomena.

3.4.9. Canonical second quantization

The fundamental result of the foregoing is that by using the particle complex structure the negative frequency solutions of the Dirac equation have positive energy, and that the physical fields are the creation and annihilation operators defined with respect to the natural complex structure.

Whilst it is rather contrary to the spirit of the geometric quantization, by which this result was obtained, I shall now consider the theory obtained by a canonical second quantization, formulated in terms of the two classes of creation and annihilation operators on the Fock space over the two complexifications of the classical phase space. The geometric quantization tells us that the complexified classical solution manifold is identified with the 1-particle[16] subspace, so this procedure is assured to make

[16] It might be better to speak of the "two-particle" or "particle - antiparticle" theory, since we shall always have

sense. At the same time we know that the standard formalism can be obtained from a modified second quantization (see (1.4.4)); one must interpret the annihilation operator for negative frequency states as an antiparticle creation operator, and normal order the resulting expressions. With the interpretation of anti-particles as negative frequency states advanced in the foregoing, and the form of the annihilation operator defined with respect to the natural complex structure, it is obvious that the standard theory is a canonical second quantization performed with respect to the *natural* complex structure, normal ordered with respect to the *particle* complex structure. That is:

The standard theory is a canonical second quantization defined with respect to J_N with Fock space $\mathfrak{F}(M_N)$, normal ordered with respect to the action of the fields on $\mathfrak{F}(M_P)$.

On the other hand using the particle complex structure the negative frequency solutions are directly interpreted as positive energy antiparicles. The canonical second quantizaton using this complex structure should give us the standard theory directly. That is[17]:

The standard theory is the canonical second quantization defined with respect to J_P with Fock space $\mathfrak{F}(M_P)$.

The most direct way of establishing these claims is to derive expressions for the various kinematic observables of the free theory. This is quite easy to do; in the space available we shall only establish the implied equivalence of the two ways of setting up the theory. As expected, the equivalence will only apply to observables which do *not* connect positive and negative frequency states.

in mind the Hilbert space of both positive and negative frequency solutions.

[17] This statement is not true of all the local quantities defined in the standard formalism, even in the free case; we shall establish the limitations of this formulation in (3.4.12), (3.4.11).

We treat the Dirac field first. The canonical second quantization $d\Gamma_P$ over M_P for such observables on M_P is given by (*cf.* Eq.(56) (1.3.4)):

$$\Gamma_P(\mathfrak{X}^P) = \sum_{ij} a^*(\psi_i)\langle\psi_i, \mathfrak{X}^P\psi_j\rangle_P a(\psi_j) \tag{61}$$

where $\{\psi_i\}$ is an orthonormal basis[18] in M_P, and $\mathfrak{X}^P$ is the self adjoint operator on M_P which generates the unitary group U_g^P on M_P corresponding to the generator X of the orthogonal group on M. On the other hand the normal ordered canonical second quantization over M_N will define the observable:

$$:d\Gamma_N(\mathfrak{X}^N): = :\sum_{i,j} b^*(\psi_i)\langle\psi_i, \mathfrak{X}^N\psi_j\rangle_N b(\psi_j): \tag{62}$$

where we must express the action of the b's on $\mathfrak{F}(M_P)$ before we can carry out the normal ordering.

The equivalence of the two expressions follows immediately. Using Eq.(48) we can write Eq.(62) as

$$:d\Gamma_N(\mathfrak{X}^N): = \sum_{i,j}\left[a^*(\psi_i^+)a(\psi_j^+)\langle\psi_i^+, \mathfrak{X}^N\psi_j^+\rangle_N - a^*(\psi_j^-)a(\psi_i^-)\langle\psi_i^-, \mathfrak{X}^N\psi_j^-\rangle_N\right] \tag{63}$$

because by assumption $\langle\psi_i^+, \mathfrak{X}^N\psi_j^-\rangle_N$ vanishes (note carefully the order of the indices i, j; the minus sign is due to the normal ordering). Since from Eq.(46):

$$\langle\psi^+, \psi^{+\prime}\rangle_N = \langle\psi^+, \psi^{+\prime}\rangle_P, \quad \langle\psi^-, \psi^{-\prime}\rangle_N = \langle\psi^{-\prime}, \psi^-\rangle_P$$

and from Eq.(36)we have that:

$$\mathfrak{X}^N = -J_N J_P \mathfrak{X}^P ,$$

we conclude:

$$\begin{aligned}\langle\psi_i^-, \mathfrak{X}^N\psi_j^-\rangle_N &= -\langle\psi_j^-, \mathfrak{X}^P\psi_i^-\rangle_P \\ \langle\psi_i^+, \mathfrak{X}^N\psi_j^+\rangle_N &= \langle\psi_i^+, \mathfrak{X}^P\psi_j^+\rangle_P .\end{aligned} \tag{64}$$

In this way we obtain:

$$:d\Gamma_N(\mathfrak{X}^N): = \tag{65}$$

$$\sum_{i,j}\left[a^*(\psi_i^+)a(\psi_j^+)\langle\psi_i^+, \mathfrak{X}^P\psi_j^+\rangle_P + a^*(\psi_j^-)a(\psi_i^-)\langle\psi_j^-, \mathfrak{X}^P\psi_i^-\rangle_P\right]$$

that is, Eq.(61).

For the complex scalar field, we have Eqs.(61) and (62) as above, but the normal ordering introduces no change of sign

[18] The summation is over both positive and negative frequency solutions, of course.

in the equation corresponding to Eq.(63). We *do* obtain the same interchange in the indices on the RHS of Eq.(63):

$$:d\Gamma_N(\mathfrak{X}^N): = \sum_{i,j}\left[a^*(\varphi_i^+)a(\varphi_j^+)<\varphi_i^+,\mathfrak{X}^N\varphi_j^+>_N + a^*(\varphi_j^-)a(\varphi_i^-)<\varphi_i^-,\mathfrak{X}^N\varphi_j^->_N\right] \tag{66}$$

But now from a comparison of Eq.(31) with Eq.(34) we have:

$$<\varphi^+,\varphi^{+\prime}>_N = <\varphi^+,\varphi^{+\prime}>_N$$
$$<\varphi^-,\varphi^{-\prime}>_N = -<\varphi^{-\prime},\varphi^->_P \tag{67}$$

and the second of these cancels the change in sign of $\mathfrak{X}^N$ acting on the negative frequency states. That is:

$$<\varphi_i^-,\mathfrak{X}^N\varphi_j^->_N = <\varphi_j^-,\mathfrak{X}^P\varphi_i^->_P$$
$$<\varphi_i^+,\mathfrak{X}^N\varphi_j^+>_N = <\varphi_i^+,\mathfrak{X}^P\varphi_j^+>_P \tag{68}$$

so that we obtain in place of Eq.(66):

$$:d\Gamma_N(\mathfrak{X}^N): = \sum_{i,j}\left[a^*(\varphi_i^+)a(\varphi_j^+)<\varphi_i^+,\mathfrak{X}^P\varphi_j^+>_P + a^*(\varphi_j^-)a(\varphi_i^-)<\varphi_j^-,\mathfrak{X}^P\varphi_i^->_P\right] \tag{69}$$

which is just the scalar version of Eq.(61). We have a very clear understanding of the fact that, whereas in both bosonic and fermionic theories the negative energy states are reinterpreted as positive energy states in exactly the same way (by the use of the "correct" complex structure), in the canonical second quantized theory the indefiniteness of the natural form $<.,.>_N$ introduces a sign change[19] provided in the fermionic theory by the use of anticommutators. We also see, that the normal ordering is essentially a device to correct for the "wrong" choice of complex structure. This is true for *both* the scalar and spin 1/2 fields; in the scalar case it is not quite so apparent.

We now consider the number and charge operators. There is an interpretation of the number operator, which we have not as yet introduced, which is that the (total) number operator is the generator of (global) *gauge transformations* of the Fock

[19] We earlier referred to this circumstance in the context of the equivalence of the classical scalar field theory with the 1-particle theory - that there is no contradiction between the positivity of the field energy and its indefiniteness in the 1-particle theory (see the discussion following Eq.(53)(1.4.4)).

space. It is, in fact, the second quantization of the generator of global gauge transformations of the one-particle space M_P, which is defined by the unitary operators

$$U_\theta = e^{-J_P\theta}$$

and hence has generator given by the unit operator on M_P (strictly speaking, we should write $U_\theta = e^{-J_P\theta\mathbb{I}^P}$, with $\mathbb{I}^P$ the unit on M_P, in keeping with our notation $\mathfrak{X}^P$, $\mathfrak{X}^N$). The number operator for a finite dimensional subspace $B \subseteq M_P$ spanned by the basis vectors $\{\psi_k\}$, $k = 1,..,N$ with projection operator $E(B)^P$ is the second quantization of $E(B)^P$, and $E(B)^P$ generates the gauge transformations $\psi \longrightarrow e^{-J_P\theta}\psi$ on $\psi \in B$, and the identity transformation on $\psi \in B$ (the orthogonal complement of B in M_P). Explicitly, we then have the unitary operators

$$U_\Theta^B = e^{-J_P\theta E(B)^P}$$

The second quantization can be written:

$$d\Gamma_P(E(B)^P) = N_B = \sum_{i,j} a^*(\psi_i)\langle\psi_i, E(B)^P\psi_j\rangle_P a(\psi_j) \qquad (70)$$

or when B is defined by the projection of test functions with support in $S \subseteq \mathbb{R}^4$ (in which case it is no longer finite dimensional)

$$N_S = \int_S a^*(x)a(x)\, d^4x. \qquad (71)$$

The integrand (the local number density operator) generates *local* gauge transformations in the Fock space[20].

The total *charge* operator Q is, on the other hand, the generator of global gauge transformations with respect to the *natural* complex structure. All of the foregoing applies to the charge operator; in particular, the charge density operator is the generator of *local* gauge transformations in the Fock space (but with respect to the natural complex structure). Because of the relationship (Eq.(36)(1.3.4)):

$$\Gamma(U)a(\psi)\Gamma(U)^{-1} = a(U\psi)$$

(and similarily for a^* and the b's), and because for the

[20] These remarks are for orientation only; I have not defined the point fields, nor shall I attempt to study the charge and number densities directly.

total number and charge operators U is given by multiplication by a complex number, we were able to write $a(U\psi)$ *etc.* in the two cases as in Eq.(41) *et seq.* (that is, using the appropriate (anti)linearity properties of the creation and annihilation operators). We now write the total operators explicitly; in terms of the natural complex structure, Q is the normal ordered second quantization of the identity $\mathbb{I}^N$ on M_N, *i.e.*

$$Q = :d\Gamma_N(\mathbb{I}^N): \tag{72}$$

it is therefore (Eq.(36)) also given by

$$Q = d\Gamma_P(-J_N J_P \mathbb{I}^N) \tag{73}$$

on M_P generating the unitary transformations:

$$\Gamma_P(U_\theta) = e^{-J_P\theta Q}.$$

In summary, the one-particle charge and number operators acting on M_N are therefore $\mathbb{I}^N$ and $-J_N J_P \mathbb{I}^P$ respectively. Acting on M_P they are $-J_P J_N \mathbb{I}^N$ and $\mathbb{I}^P$ respectively. Since the canonical second quantization defined by the particle complex structure preserves positivity (which is destroyed by normal ordering with respect to some other complex structure) it is the number operator, and not the charge, which is positive definite. Using the natural complex structure, with no normal ordering, one would obtain the charge as positive definite and the number indefinite.

We now consider charge conjugation. For the Dirac field, we may exploit the explicit form of the projection operators acting on the momentum space representation:

$$P^{\pm} = (m \pm \gamma^\mu p_\mu)/2m. \tag{74}$$

We have not introduced the plane wave expansions in the present context; however we do not need them explicitly. It is enough to recall that $\gamma^0=\gamma^{0t}$, $\gamma^0\gamma^{\mu *}\gamma^0 = \gamma^\mu$, and that $\mathcal{C}\gamma^\mu\mathcal{C}^{-1} = -\gamma^{\mu t}$ (Eqs.(60), (55), (57) of (1.4.4)), together with the 1-particle charge conjugation operator:

$$\mathcal{C}: \psi \longrightarrow \psi^c = \mathcal{C}\gamma^{0t}\bar{\psi}$$

(Eq.(59),(1.4.4)). From Eqs.(74), (39) we deduce that

$$J_P = i\gamma^\mu p_\mu/m$$

so that $(J_P\psi)^c = \mathcal{C}\gamma^{0t}\overline{J_P\psi} = -i\mathcal{C}\gamma^{0t}(\gamma^\mu p_\mu)^{*t}\gamma^{0t}\mathcal{C}^{-1}\psi^c$. From the foregoing we conclude:

$$(J_P\psi)^c = -i\mathfrak{C}(\gamma^\mu p_\mu)^t\mathfrak{C}^{-1}\psi^c = i\gamma^\mu p_\mu\psi^c = J_P\psi^c$$

so that $\mathfrak{C}$ is in fact linear with respect to the particle complex structure. It follows that in fact:

$$\mathfrak{C}\, J_P\mathfrak{C}^{-1} = J_P. \qquad (75)$$

Since further $\mathfrak{C}$ interchanges positive and negative frequency solutions we conlude that its canonical second quantization is what we previously referred to as $\mathfrak{C}_F$ (see (3.3.8)). That is, since $\Gamma_P(U)a(\psi)\Gamma_P(U) = a(U\psi)$ it follows that with $U = \mathfrak{C}$, $\Gamma_P(\mathfrak{C})$ will interchange particle annihilation operators with anti-particle annihilation operators (and likewise for the creation operators). Therefore

$$\mathfrak{C}_F = \Gamma_P(\mathfrak{C}).$$

There is no longer any apparent contradiction between the anti-linearity of $\mathfrak{C}_F$ at the level of the fields, and its linearity at the level of the particle and anti-particle Hilbert spaces noted previously. Neither $\mathfrak{C}_F$ nor $\mathfrak{C}$ is antilinear at either level. For the scalar case exactly the same conclusion holds; the charge conjugation is then just:

$$\mathfrak{C}:\ \varphi \longrightarrow \bar{\varphi}$$

so that:

$$J\varphi \longrightarrow \overline{J\varphi} = \overline{i\varphi^+ - i\varphi^-} = -i\varphi^- + i\varphi^+ = J\varphi$$

as required.

Finally, we see that there is no longer any problem at the one-particle level with the charge-current vector. In the standard formalism the field counterpart changes sign due to the normal ordering[21]; using the particle complex structure we do not need to normal order and we must obtain the sign change at the level of the 1-particle theory. It is clear that we have problems using the natural complex stucture because the total charge operator is just $\mathbb{N}^N$; it is obvious that no change in sign is possible under a unitary or anti-unitary transformation. On M_P it is $-iJ_P$ and we have from Eq.(75) that:

$$-iJ_P \longrightarrow \mathfrak{C}(-iJ_P)\mathfrak{C}^{-1} = i\mathfrak{C}(J_P)\mathfrak{C}^{-1} = iJ_P.$$

[21] That is, for fermion fields. We have already noted that the normal ordering plays a more obvious role in correcting for the "wrong" complex structure in fermion theories.

Note that in M_P the (positive energy, negative frequency) positron states are the charge conjugate of the (positive energy, positive frequency) electron states. In conventional accounts of the 1-particle theory, which considers M_N as the Hilbert space, the (positive energy, positive frequecy) positron states are supposed to be the charge conjugate of the *negative* energy, negative frequency electron states. The latter are positive energy positron states in terms of J_P; taking their charge conjugate will on the contrary give the *electron* states.

In conclusion we see that we may obtain every field operator of the standard formalism which does not connect positive and negative frequency states (*i.e.* which does not lead to pair creation and annihilation) as the canonical second quantization of a 1-particle theory, with no additional assumptions foreign to the canonical theory of NRQM. Concerning the local correspondence that we have in NRQFT, see (3.4.12).

3.4.10. Postscript: the Dirac negative energy sea.

It is apparent that the hole theory, using the operators defined with respect to the natural complex structure and with the simple expedient of replacing the creation and annihilation operators $a^*(\psi)$, $a(\psi)$ by the operators $a^*(\psi^+)+a(\psi^-)$, $a(\psi^+)+a^*(\psi^-)$, and then normal ordering, precisely corrects for the use of the natural, rather than the particle, complex structure in the canonical second quantization.

As we concluded in (1.4.4), we see that the hole theory is only used explicitly to motivate this replacement. The hole theory in its turn rests on a single assumption: that the vacuum is the state with all negative frequency states occupied (the "negative energy sea"). Since the complex structure directly defines the vacuum state it is no surprise that this amounts to a change in the definition of

the complex structure, that is the hole theory is a way of implementing the transition from the natural to the particle complex structures (the pre-hole theory vacuum is of course defined with respect to the natural complex structure). We may therefore say that the negative energy sea is what the *usual* Fock vacuum (the no-particle state) defined with respect to the *particle* complex structure looks like from the point of view of the no-particle state defined with respect to the *natural* complex structure.

That the former (the no-particle state defined with respect to the particle complex structure) can be described specifically as the state with all negative energy states occupied (from the point of the no-particle state defined with respect to the natural complex structure) is illuminated by the following theorem[22]. Consider a *finite* (even) dimensional real vector space V and define $\Lambda^p(V)$ as the space of all p-forms on V (that is, skew symmetric tensor products of 1-forms on V). We introduce a complex structure J on V and define $\Lambda^p(V_J)$ similarily (note that when V is infinite dimensional $\mathfrak{F}(V_J)$ can be defined as $\oplus_{p=0}^{\infty}\Lambda^p(V_J)$). Then it is a theorem that $\Lambda^p(V_J)$ is isomorphic to $\Lambda^{n-p}(V_{-J})$, where n is the dimensionality of V (this is the complex version of the isomorphism between $\Lambda^p(V)$ and $\Lambda^{n-p}(V)$ provided by the Hodge * operator). The significance of this result is that $\Lambda^0(V_J)$ is one dimensional, representing the vacuum state with complex structure J; it is therefore isomorphic to $\Lambda^n(V_{-J})$, which is also one-dimensional (think of the uniqueness of the volume 4-form on $\mathbb{R}^4$), which corresponds to the negative energy sea (all states occupied) defined with respect to the reversed complex structure -J. Consider V_J as the Hilbert space of negative frequency states, with complex structure J given by multiplication by -i (this is the particle complex structure); then V_{-J} is the complex conjugate space, or the Hilbert space with complex structure -J given by

[22] The proof is quite straightfoward, proceeding from (5.5.120) and (4.2.82) of Lichnerowicz [1976].

multiplication by +i (the natural complex structure). In fact to make this argument precise we must take n as infinite, in which limit the theorem does *not* hold; this is connected with the fact that the Dirac negative energy sea cannot be defined in Fock space (this vacuum, like the first Dirac vacuum discussed in (1.2.4), defines a non-Fock representation; see (3.5.1)).

The historical commentary on the hole theory (*e.g.* Pais [1986], Wightman [1972]) tends to the view that the theory cast in standard form (field theory + correct particle interpretation via the plane wave expansion + normal ordering), that is essentially following Heisenberg's treatment of [1934], is in all respects superior to the hole theory:

> The way was open to write the theory of electrons and positrons in a form completely symmetric under the exchange of particle and antiparticle, a form in which the infinite sea of negative energy electrons has vanished from the theory except as a poetic description of the prescription for forming the electromagnetic current...the essential change in the formula for (the field) as compared with the pre-hole theory expression is the appearance...of the positron creation operator...in the pre-hole theory expressions there would have been an annihilation operator for negative energy electrons. The prescription for the electric current referred to above is
> $j^{\mu}(x) = :\psi\ \gamma^{\mu}\psi:(x)$
> the subtracted term was thought of as the contribution from the negative energy sea. Today we have become reconciled to (such subtractions)... .
>
> The new way of looking at the Dirac equation did not arouse the widespread satisfaction one might have expected. Dirac himself was inclined to work directly with the infinite sea ....Pauli was dissatisfied with the prescriptions of hole theory, regarding them as ugly and artificial. He coined the derisory term "subtraction physics" to describe hole theory. With Weisskopf he developed a quantum field theory of a scalar field describing charged spin zero particles. It, like the hole theory, predicted such phenomena as pair production by an external field. For reasons that now appear a bit incomprehensible since, as Pauli himself showed, the theories can be constructed in parallel, he regarded the treatment of subtractions in the Pauli-Weisskopf theory as natural, and he dubbed it the *anti-Dirac* theory. Nevertheless, it was clear that a revolution had taken place. To describe the electron with full precision, one used the Dirac equation, but the Dirac equation for a

field. (Wightman [1972 p.100-102]).

I believe that the theory did not arouse "the widespread satisfaction one might have expected" because in the early 30's, but not subsequently, physicists were more concerned with the canonical structure of NRQM. I also believe that the present theory vindicates this concern: there are profound connections between the free field theory and the canonical second quantized theory, and the hole theory is a physical interpretation of the procedure by which one defines the physical fields as creation and annihilation operator with respect to the natural complex structure, whilst defining the particle interpretation by means of the particle complex structure. The theorem discussed above is what makes this physical interpretation possible, but there is no such theorem for the symmetric case. But one does just the same thing in the bose theory, so one must consider this physical interpretation unsatisfactory.

3.4.11. On linear interactions: heuristics.

The physical fields (natural creation and annihilation operators) are what get coupled to external fields. At the same time, the natural complex structure is given by the classical field theory and is respected by the Lagrangian (in the sense that the interaction Lagrangian is N-gauge invariant). And further, the classical interacting theory is covariant and local on spacetime and the fields are causal with respect to spacetime. The latter criterion excludes coupling to the particle creation and annihilation operators, for q-space is not a G-space for the ILG and the particle creation and annihilation operators are q-causal only, because J_p is non - c-local.

This is *one* reason for the gulf opened up between RQFT and NRQFT, for in the latter theory one couples directly to the particle creation and annihilation operators[23]. What if one

[23] Since this coupling is bilinear in these operators no pair

abandons causality, and proceeds as in NRQFT? *If in* Eq.(61) *the operator* $\mathfrak{X}^P$ *connects positive and negative frequency states, we will obtain processes in which particles convert directly to antiparticles.* The reason is obvious: the canonical theory will give us gauge invariance with respect to the *particle* complex structure (equivalently: bilinearity in particle creation and annihilation operators), and thereby ensure conservation of total particle number, but *not* the charge.

It is important to emphasise that a general expression of the form:

$a^*(\varphi)<\varphi, \mathfrak{X}^P\psi>_P a(\psi)$

is *not* N-gauge invariant when $\mathfrak{X}^P$ connects positive and negative frequency states; for then (in the notation of Eq.(41)) one has under J_N: $\varphi \longrightarrow e^{i\theta}\varphi$ (and similarily for ψ) a part:

$a^*(e^{i\theta}\varphi^+)<e^{i\theta}\varphi^+, \mathfrak{X}^P e^{i\theta}\psi^->_P a(e^{i\theta}\psi^-) = a^*(e^{J_P\theta}\varphi^+) \aleph\, a(e^{-J_P\theta}\psi^-) =$
$= e^{2i\theta}\, a^*(\varphi^+) \aleph\, a(\psi^-)$

(where I have written $\aleph$ for $<e^{i\theta}\varphi^+, \mathfrak{X}^P e^{i\theta}\psi^->_P$). All would be well if $\aleph$ reduced to $e^{-2i\theta}<\varphi^+, \mathfrak{X}^P\psi^->_P$. However, this is not generally true; whilst:

$$<e^{i\theta}\varphi^+, e^{i\theta}\psi^->_P = <e^{J_P\theta}\varphi^+, e^{J_P\theta}\psi^->_P = e^{-2i\theta}<\varphi^+, \psi^->_P$$

(recall that $<.,.>_P$ is only sesquilinear with respect to J_P), because $\mathfrak{X}^P$ connects positive and negative frequency parts, it cannot commute with the complex structure J_P; that is $e^{-iJ_P\theta}\, \mathfrak{X}^P e^{iJ_P\theta} \neq \mathfrak{X}^P$.

We now see that there is *another* reason for the gulf between relativistic and non-relativistic quantum theory. For linear interactions of this kind, *the canonical second quantization does not yield an* N*-gauge invariant theory.* Indeed, just because it remains P-gauge invariant, it *cannot* also be N-gauge invariant. N-gauge invariance means we must define the observables by bilinear expressions in the *natural*

creation and annihilation processes are possible. If it were not bilinear, the coupling would not be P-gauge invariant.

creation and annihilation operators. By exactly the same reasoning as above, it follows that such terms will *not* be P-gauge invariant. The theory conserves charge at the expense of particle number. For a linear perturbation $\mathfrak{X}^N$ of this form, we find, when making the transition from Eq.(61) to Eq.(63), additional terms of the form:

$$:d\Gamma_N(\mathfrak{X}^N): \; = \sum_{i,j}\left[a^*(\psi_i^+)a^*(\psi_j^-)\langle\psi_i^+, \mathfrak{X}^N\psi_j^-\rangle_N \; - \; a(\psi_i^-)a(\psi_j^+)\langle\psi_i^-, \mathfrak{X}^N\psi_j^+\rangle_N\right]$$

which represent pair creation and annihilation. When $\mathfrak{X}_N$ and J_P fail to commute, one simultaneously learns that $\langle\psi_i^-, \mathfrak{X}_N\psi_j^+\rangle_N$ no longer vanishes and that J_P is no longer preserved by the evolution; the particle complex structure rotates. This is just the situation in which q-local densities have no sensible physical interpretation, because the non-commutativity of $\mathfrak{X}_N$ with J_P implies that $\mathfrak{X}_P$ connects positive and negative frequency solutions.

On the other hand the natural complex structure commutes with every c-local observable, and is itself c-local. The theory developed in this way is covariant. The particle interpretation must still be defined by picking a particle complex structure, of course, but this will now vary with the time and the theory is *not* P-gauge invariant.

In these two ways we have understood the meaning of the criteria, established in Section 3.3, that the physical fields must satisfy: *covariance, c-causality, and* N-*gauge covariance.*

The explanation for the fact that only *pair* creation and annihilation processes are possible follows from N-gauge invariance and P-gauge non-invariance. The only way for particle number to change, with no change in the total charge, is through pair creation and annihilation. This heuristic is not perhaps as satisfactory as that of the hole theory, but has the virtue that it applies equally to the bosonic theory.

It is perhaps the Stueckelberg-Feynamn interpretation which provides the clearest heuristic as to why only pair creation

and annihilation processes are possible (namely that the number of particle world lines is invariant and no world line terminates at a point; the reversals in time correspond to pair creation and annihilation events). This interpretation is beyond the scope of this thesis, but the following observation appears fundamental: in interpreting antiparticle states as negative frequency states evolving backwards in time, one is effectively reversing the Hilbert space complex structure for these states. There is therefore a close connection between the Stueckelberg-Feynamn interpretation and the present theory[24]. I suggest that the present theory is essentially the transcription of the Stueckelberg-Feynman theory into the canonical framework of NRQM, just as it is the transcription of the hole theory into the canonical framework.

3.4.12 The local equivalence.

It is clear from the foregoing that the canonical second quantization of arbitrary multiplicative functions on spacetime do not have a sensible physical interpretation; there is no spacetime local correspondence. But we have already seen (3.3.7) that we can, and should (to define the Schrödinger construction of (1.3.3)), formulate a local equivalence between *q-local* 1-particle operators (that is finite polynomials in multiplicative and derivate transformations of functions on position space), and q-local densities in the position space fields. We now see (from (3.4.7)) that the map $f,g \longrightarrow Cf + iC^{-1}g$, $Cf-iC^{-1}g = \chi,\zeta$ (describing a 1-particle state χ and the anti-particle state ζ) induces a correspondence[25] between operators on the position space Hilbert space (of pairs χ,ζ) and 1-particle operators on M. When the former are given as linear maps $\mathcal{O}$:

[24] There may be an expression of the Stueckelberg-Feynman heuristic in the present approach. I have not found it, however.

[25] I shall not here attempt to define this correspondence precisely.

$(\chi,\zeta) \longrightarrow (\mathcal{O}^{+}\chi,\ \mathcal{O}^{-}\zeta)$ where $\mathcal{O}^{\pm}$ are operators on $L^2(R^3,d^3x)$ then $\mathcal{O}$ automatically commutes with J_P: $\chi,\zeta \longrightarrow i\chi,\ -i\zeta$. Further, by consideration of the inner product on position space (Eq.(52) one expects that $\mathcal{O}$ cannot then connect particle to antiparticle states; it must correspond to an *even* operator on M (in the sense of Schrödinger). We suppose that such $\mathcal{O}$ can all be defined in terms of the commuting ring of $P^{\pm}$ (as a sub-algebra of $\mathbf{B}(M_P)$, the von Neumann algebra $\{P^{\pm}\}'$ (where $\{P^{\pm}\}$ is the algebra generated by $P^{\pm}$). By construction $\mathcal{O}$ in $\{P^{\pm}\}$ will satisfy the equivalence of Eqs.(61) and (62), but of course if $\mathcal{O}^{\pm}$ are local in position space acting on L^2, $\mathcal{O}$ acting on M_P will not be local (and hence on M_N or M - these spaces are pointwise the same). Clearly we can write the canonical creation and annihilation operators on $\mathfrak{F}(M_P)$ as *e.g.* $a(\varphi) = a(\chi) + a(\zeta)$ so that the second quantization $d\Gamma_P(\mathcal{O})$ can be written as in Eq.(61) but using the position space states, and in a suitable limit, when $\mathcal{O}^{\pm}$ are q-local (i.e. local in their action on the Cauchy data $(Cf,C^{-1}g)$), in the form of Eq.(23) (1.3.3). Of course from the form of the inner product on position space we also know that the position space creation and annihilation operators will commute when the supports of $Cf,C^{-1}g$ have vanishing intersection; they are therefore q-causal. Evidently we have an exact local equivalence (between the position space 1-particle theory and position space q-local densities in the fields, as indicated by Eq.(43) (3.3.7)); the difference is we have a natural structure, relating the 1-particle system to the particle -anti-particle system, and characterising position space in terms of the representation of the Cauchy data for the system so that the complex structure acts locally. Automatically the inner product determining the (anti)commutators of the creation and annihilation operators become causal relative to this representation.

The synthetic power of the present formalism is remarkable; and we may finally conclude that in this non-covariant sense (i.e. working throughout on position space) the relationship of RQFT to RQM is precisely the same as in the

non-relativistic theory, with a complete q-local correspondence for all q-local operators in $\{P^{\pm}\}'$. We could at this stage attempt to introduce interactions by q-local couplings of this form.

Why would this theory be "wrong"? The answer can only be such q-local external potentials do *not*, in fact, exist[26].

I have already discussed the implications for our view of what count as "true" locality and "true" local densities in the fields in (3.2.7) and (3.3.8). The c-locality, and N-gauge invariant densities, are the criteria used to define interactions; these desiderata make no sense from a purely particle point of view, where they appear as generators of transformations which violate P-gauge invariance and are highly non-local. From the point of view of the particle-antiparticle structure of the field this is the "true" nature of dynamics; presumably from the point of view of the charge, with no information on the particle-antiparticle content of the field, this evolution is locally determined[27].

[26] However for time-independent potentials which are reasonably smooth one can modify the definition of the particle complex structure (essentially so that it commutes with the total Hamiltonian). In this way one can generalize the forgoing to quite a wide class of time-independent interacting fields (c-number external couplings). See Bongaarts [1972]. In this sense the dynamics must be built into the quantization. Such theories are called quasi-free, and it is plausible that they describe the actual phenomenology of particle detection.

[27] I would like to indulge in the following speculation. Considering the physical fields now appear as canonical creation and annihilation operators defined by the natural complex structure, with the charge as generator of gauge transformations, one has a canonical formulation of the intuitive idea that the physical fields are the creation and annihilation operators for a single unit of charge. The canonical second quantized theory set up in this way (without anywhere introducing the particle complex structure) c-locally corresponds to the covariant 1-particle theory, with all the attendent problems of negative energy states and indefinite norm. One is tempted to consider that in some sense in relativity the concept of charge plays the role of the concept of particle and the concept of charge

3.4.13. Postscript: Non relativistic quantum field theory.

We have left until last the fundamental mathematical structure with which we began; the canonical theory of NRQFT. In this theory recall that we begin from a 1-particle theory, and then form the Fock space and the canonical creation and annihilation operators, which in configuration space are the non-relativistic quantum fields.

If instead we consider the Schrödinger equation as a classical field equation and develop its symplectic geometry[28], we would define a canonical Fock representation by finding a positive compatible complex structure with respect to which the Hamiltonian is postive.

There is a natural complex structure present as in the relativistic theory (multiplication by i). The Hamiltonian is *positive* with respect to this complex structure. It is moreover c-local so that the inner product is c-local and the creation and annihilation operators, together with the Segal field, are c-causal. One could say, that the natural and particle complex structures coincide in the non-relativistic case: the distinction between the physical

density replaces the concept of probability density. This works well for probabilities associated with particle position, but it is not clear what should replace probabilities associated with energy. We cannot have probabilities for fermions, because what should be the energy density is indefinite. In some sense we need to interpret it as a charge-energy density, that is potential energy density. For fermions we have the odd situation that the potential energy density is indefinite but the charge density definite, whilst for bosons it is the other way round. It is remarkable that superposing a particle interpretation on this theory then restores the symmetry in the two theories. A logical step at this point is to try to extend the present methods to supersymmetric theories.

[28] The symplectic form is $\omega(\phi,\phi') = \frac{i}{2}\int (\bar{\phi}\phi' - \phi\bar{\phi}')d^3x$. One can also treat the same classical theory as a Clifford algebra; in that case the symmetric form is $S(\phi,\phi') = \frac{1}{2}\int (\bar{\phi}\phi' + \phi\bar{\phi}')d^3x$.

fields and the particle creation and annihilation operators vanishes in the non-relativistic theory. In this situation, an N - gauge invariant interaction Lagrangian must be bilinear in the Hilbert space complex structure as well, so that no pair creation and annihilation processes are possible. The concepts of charge and particle number are thus identified (and the charge is just a c-number multiple of the particle number; it has therefore only a single sign), and both densities are c-local. The Born interpretation is covariant, because the natural complex structure is c-local.

Incidentally, we have met another field theory where there is only a single complex structure present: that is, the real scalar field, where there is only the particle complex structure. It appears that it is the real scalar field which is the closest analogue of the non-relativistic scalar field.

3.5. Non-Fock Representations

> Even if the world is infinitely complex, so that every fact consists of infinitely many states of affairs and every state of affairs is composed of infinitely many objects, there would still have to be objects and states of affairs.
>
> L. Wittgenstein

3.5.1. Introduction.

As we have remarked on several occasions, the Stone-von Neumann-Mackey uniqueness theorem does not apply to systems of infinitely many degrees of freedom. A quantum field is such a system; there are an uncountable infinity of inequivalent irreducible representations of the Weyl algebra.

Since the inception of quantum field theory, however, the Fock-Cook representation formed the basis of all its physical applications[1]; effectively the theory was *defined* in terms of the Fock-Cook representation. Of course the particle point of view, according to which particles are considered the fundamental constituents of the world, makes this strategy very reasonable; the physical fields, to be defined via a canonical second quantization as in Section

[1] There were the occassional exceptions, such as the preliminary investigation of the representation in which the states were considered functionals over classical field configurations (Jordan and Pauli [1928], cf. fn. 28, (1.4.4) and also below, (3.5.2)), and more recently the applications to quantum optics (Glauber [1963], Sudarshan [1963]), but in all these cases the unitary equivalence with the Fock-Cook representation was assumed.

3.3, are then convenient constructions for the introduction of interactions, and if there is any problem in fixing the particle representation of the fields it is because we are using too general a mathematical theory for their description.

One finds very little discusion of this issue; because of the formalist inclinations of the physics community, because of the overwhelming dominance of scattering phenomenology throughout most of the history of QFT, or because philosophical debate has been almost entirely confined to NRQM - for these and other reasons, alternatives were not explored, not even the issue: should RQFT be defined by the Fock representation? raised more than passing comment.

Here is an example. In a slightly different context (the question of the logical necessity for quantizing the electromagnetic and gravitational field) Leon Rosenfeld, in a sharply worded note, declares:

> There is a further erroneous view to which misplaced emphasis on formal aspects of quantum theory has led, and which is so wide-spread that it has even (I am sorry to say) found expression in the incriminated introductory section of the textbook quoted at the beginning of this note (Henley and Thirring [1962]). It is there asserted that the wave functions associated with material particles represent "classical matter fields", and from this patently wrong statement it is then argued that these "matter fields" should also be quantized. Actually, as ought to be well known, the scheme of quantum mechanics (in any representation) constitutes a complete incorportation of the quantum postulates in the description of the behaviour of material particles, and what is misleadingly called "second quantization" is nothing else than an equivalent formulation of this scheme with the help of convenient operators in a representation characterized by the choice of the numbers of particles in given individual states as variables. In particular, the commutation or anti-commutation relations satisfied by the annihilation and creation operators have nothing to do with the quantization of the field (they do not contain Planck's constant), but are just algebraic properties of these operators. (Rosenfeld [1963 p.355]).

Rosenfeld, more than anyone else, should count as an expert on the logico-philosophical analysis of quantum field

theory. And that is all he had to say about it[2].

On a purely particulate basis, the failure of the von Neumann uniqueness theorem can only come about if one assumes that one has an *actual infinity* of particles in the universe, for only in this way will one have an infinite number of degrees of freedom. This is not true of a quantum field theory, where the actual infinity of the degrees of freedom of the field need not be excited.

Therefore the issue - is the standard formalism *committed* to the existence of quantum fields? - bears directly on the admissability of non-Fock representations, most particularly on the general realist philosophy that one never describes a literal infinity of particles. In this context there seem to be only three[3] ways of defending a purely particulate formulation of (interacting) RQFT: the first is to appeal to the Feynman theory, the second is to use the S-matrix theory, and the third is to establish that RQFT can be rigorously defined in Fock space, and conclude on the basis of Section 3.3 that one thereby reduces the theory to a modified canonical second quantization. Against the latter, we insist that if this is possible, one should be able to develop a non-trivial theory *without* the field concept, for the field is then relegated to an auxilliary calculational device. On the basis of what we know of this equivalence, and in view of the restriction of even the hole theory and the Feynman theory to pair creation and annihilation processes only, this prospect appears so remote for systems which undergo single creation and annihilation events as to be pure fantasy.

There is one class of physical systems which we have not

[2]The inadequacy of this critique is obvious, and there seems to be no point in laboring its shortcomings.

[3]We do not consider the Dirac hole theory because this is, precisely, a non-Fock representation. We mention in passing that string theory also leads to non-Fock representations, because the classical string is a system of infinitely many degrees of freedom.

considered. Rosenfeld was not himself committed to the view that the radiation field, too, be fundamentally interpreted as a particle theory (*cf.*, in this connection, his rider that the principles of quantum mechanics are sufficient for the description of *material* particles). There are a number of ways in which the radiation field is distinguished: because the photon is not-localizable, because photons are emmitted and absorbed singly, and because the field has a classical limit with a well-defined classically observable phenomenology. The first and third are of course related, whilst the second prohibits a particle interpretation along the lines of the Feynman theory or, on the basis of the foregoing, the canonical second quantization, and therefore of any known particle theory. In fact only the electromagnetic and the gravitational fields might, on this view, be considered appropriate physical systems which might justifiedly be described by means of non-Fock representations.

So much for a strict realist philosophy based on the particle concept. The pragmatic realism espoused by working physicists (from Monday to Friday, as it were), accepts the existence of a literal infinity of *virtual* particles, and indeed the standard working interpretation of QED describes the physical vacuum as such a system. In the actual practise of physics, therefore, and quite apart from one's views on the reality or otherwise of the field itself (that is to say, purely on the basis of this pragmatic particle interpretation), one cannot actually formulate this heuristic, nor the largely figurative mathematics of QED, in Fock space. On a somewhat different basis, but once again pragmatic in character, the description of thermodynamic properties of particle systems is most frequently achieved by the assumption that the number of particles is infinite. Here, in contrast to the foregoing, this assumption is considered an idealization, that is the philosophy is instrumentalist, and again in contrast to the foregoing, the resulting theory has been developed in a rigorous way. The failure of uniqueness for the representation theory has in this field been exploited.

There are two other ways in which one is forced to take seriously the existence of non-Fock representations. The first is in connection with spontaneous symmetry breaking, and more generally with the degeneracy of the physical vacuum, and the second is in connection with gravity. In view of the close connection between the particle concept and the existence of a global spacetime symmetry group, it is very clear that gravity will pose rather special difficulties to the construction of a Fock representation, whilst the concept of vacuum degeneracy cannot be defined in a Fock representation.

In the next four sections we elaborate on the foregoing ideas, although because of their breadth and difficulty our treatment will be superficial, and four of the most important are ommitted altogether: spontaneous symmetry breaking, zero-mass fields, S-matrix theory, and the Feynman theory. There are important applications of non-Fock representations to the first of these; its ommision makes our survey incomplete as well. The last three bear on the question of whether we have to consider non-Fock representations at all: one for, and two against. Since present trends run counter to the S-matrix and Feynman theories, this neglect is at least in keeping with the times.

In (3.5.2) the differences of field and particle systems, viewed as systems of infinitely many degrees of freedom, is discussed; in the process we shall see in a little more detail what is meant by the equivalence of field and particle theory. In (3.5.3) some historical background is provided on the early recognition of the relevance of inequivalent representations to quantum physics, and in (3.5.4) we consider the way in which the same problem was encountered in another branch of physics almost twenty years later, namely quantum gravity, specifically the free field theory on a classical curved background. The problem in this context is directly linked to the question of the existence of compatible positive complex structures on the solution

manifold of linear classical fields.

Subsection (3.5.5) develops a *brief* introduction to thermodynamic applications. The rest of this section develops a philosophical account of non-Fock representations of this kind, most particularly in application to the measurement problem.

In this section we use natural units $\hbar = c = 1$.

3.5.2. Fields and particles; isomorphic representations and differences in interpretation.

If one has a classical dynamical theory, then, in a way which is made precise by the geometric quantization, one knows intuitively what the Hilbert space of the corresponding quantum theory is going to be, and at least for the kinematic observables one can define a class of operators which generate the associated one-parameter groups. The Hilbert space is, roughly speaking, a space of well-behaved functions on a subspace of the classical phase space (by well-behaved we mean an L^2 space with respect to some measure, and the subspace in question has half the dimensionality of the classical phase space), or else it is a space of entire functions on the classical phase space, defined with respect to a suitable complex structure.

In (3.4.2) we developed a field quantization using the latter of these two strategies; we also indicated how this quantization may be effected for the 1-dimensional simple harmonic oscillator. The Fock representation in particular was introduced, from a purely field theoretic point of view, through a power series expansion of the entire functions on the classical phase space, equipped with a suitable complex structure. We now consider once more the interpretation of the Fock representation within field theory, from the point of view of understanding better the circumstance, that whereas from the field point of view one has a system of infinitely many degrees of freedom, from the particle point of view arbitrary states always have finite particle number

in the Fock representation. We are no longer concerned with specific details of any particular field theory; what follows is quite general.

To this end we consider once more the physical interpretation, that the field is an infinite collection of harmonic oscillators, that the particle interpretation comes about because each of these oscillators, when quantized, may only take on discrete energies, and that their excitation number determines the number of particles present in that state (*i.e.* as defined by the frequency of the oscillator). However this time we shall not use the holomorphic representation, in which the states of the quantum field are entire functions on the classical solution manifold. Our discussion in (3.4.2) on this was entirely heuristic: in finite dimensions (that is, with the elements of the classical solution manifold replaced by vectors in a finite dimensional complex vector space V) such an entire function can be written as a Taylor series; each term in the polynomial is a product of the form (*cf.* Eq.(8), (3.4.2))

$$C^{n}_{ab\ldots c} z^{a} z^{b} \ldots z^{c} \qquad (1)$$

in which z^{a} is the ath component of z with respect to a symplectic frame on the finite dimensional vector space V, and $C^{n}_{ab\ldots c}$ is a complex constant symmetric in a,b,...,c. Since each $z \in V$ is supposed to define an element of the classical solution manifold, the field - many-particle equivalence rests on the correspondence between elements of the classical solution manifold, equipped with the particle complex structure, and quantum one-particle states. Particle identity here follows as a consequence of the fact that the n-particle component of an arbitrary state must be of the form Eq.(1), in which the order of the z's has no meaning.

Elegant though this formulation be, the physical intuition, the picture of the field as an infinite collection of harmonic oscillators (each of which defines a set of particles, the number of which corresponds to the excitation level of the oscillator) lies obscured in the choice of a symplectic basis for V, whilst the quantization of each such oscillator is implicit (and hence not clearly distinguished)

in the use of entire functions on V. We may proceed differently; we can begin *ab initio* with the field as an infinite collection of quantized 1-dimensional harmonic oscillators, and obtain the Fock representation from the Hilbert space of this system. This heuristic stems from Debye [1910] and the approach to QFT of (1.1.4).

We begin from the canonical L^2 representation for the one-dimensional harmonic oscillator, that is the Hilbert space $\mathfrak{h} = L^2(\mathbb{R}, d\mu)$. The state space of the field viewed as an infinite collection of such oscillators is then of the form $\mathcal{H} = \mathfrak{h}\otimes\mathfrak{h}\otimes\ldots\otimes\mathfrak{h}\otimes\ldots$. which for Lebesgue measure would lead to the limit, as $n \longrightarrow \infty$, of the Hilbert space $L^2(\mathbb{R}^n, dx^n)$. Unfortunately there is no Lebesgue measure in this limit; for this reason we must choose some other measure which makes sense on the space $\mathbb{R}^n$ in the limit $n \longrightarrow \infty$, which is to say we must work with a measure which goes over to a measure on what becomes essentially a function space.

This problem has long been of interest within probability theory, most specifically in connection with Brownian motion and the general theory of random processes. It also arises in quantum physics through the Feynman path-integral formulation; the strategy which has proved most fertile there, and which leads to essentially a theory of random processes, is the replacement of the time parameter by a pure imaginary parameter (the Wick rotation), which in the one-particle non-relativistic case converts the Schrödinger equation into the Fourier equation of heat conduction (Nelson [1964]). A similar strategy in the relativistic case leads directly to the Euclidean QFT of Symanzik [1964], Nelson [1973] and others (see Simon [1974] for a general review).

The essential point in the present context is an early insight of Segal [1954], that initiated the development of this connection between quantum theory and probability theory, namely that that the Fock space representations correspond specifically to *Gaussian* measures on an underlying function space (namely Wiener measure) and thereby to Gaussian random variables. In these

circumstances the free boson theory is essentially equivalent to a species of classical stochastic processes, and such field theories are in a certain sense embedded in classical statistical mechanics.

It should be clear that there are profound and far-reaching consequences of the use of Gaussian measures at this point of our construction, but for now we can make their introduction appear quite innocuous by the observation that $L^2(\mathbb{R}, d\mu)$ and $L^2(\mathbb{R}, d\mu')$ are isomorphic when μ and μ' are mutually equivalent (*i.e.* have the same null sets) and if $d\mu$ is dx then $d\mu' = \phi(x)dx$ is mutually equivalent if ϕ is nowhere zero. For convenience we choose the Gaussian measure $d\mu' = e^{-x^2/2}\,dx$. The function

$$\phi(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/4}$$

is the ground state in $\mathcal{H} = L^2(\mathbb{R}, dx)$ with zero eigenvalue for the Hamiltonian

$$H = (p^2+q^2)/2 - 1/2$$

(in which the zero-point energy is subtracted). If we define the isomorphism $U: \mathcal{H} \longrightarrow \mathcal{H}'$ by

$$(U\psi)(x) = \psi(x)/\phi(x)$$

we see that the ground state in the unitarily equivalent Hilbert space $\mathcal{H}' = L^2(\mathbb{R}, d\mu') = L^2(\mathbb{R}, |\phi|^2 dx)$ is defined by $\phi'(x) = 1$. The operator q is unchanged in this new representation if it is multiplicative on $\mathcal{H}$.

In this form the heuristic discussion which follows may be made rigorous; in the limit $n \longrightarrow \infty$ the measure $|\phi|^{2n}dx^n$ becomes the Weiner measure.

It is an elementary fact that the Hermite polynomials $H_n(x)$ form an orthonormal basis for the space $\mathcal{H} = L^2(\mathbb{R}, |\phi|^2 dx)$ (hereafter we drop the primes); in particular for any $\psi \in \mathcal{H}$ we can write

$$\psi = \psi_0 + \psi_1 + \ldots$$

where $\psi_n = c\, H_n$, $c \in \mathbb{C}$. Each H_n spans the symmetrized subspace $(\mathbb{C}\otimes\ldots\otimes\mathbb{C})_S$ of $\mathbb{C}^n$ and in this way we can regard $\mathcal{H}$ as the space $\mathbb{C} \oplus \mathbb{C} \oplus \ldots \simeq \mathbb{C} \oplus \mathbb{C} \oplus (\mathbb{C}\otimes\mathbb{C})_S \oplus \ldots = \mathfrak{F}_S(\mathbb{C})$. It is

a remarkable property of Fock space that one has[4]:

Theorem 3.5.1

$\mathfrak{F}_S(\mathcal{H}_1) \otimes \mathfrak{F}_S(\mathcal{H}_2) \simeq \mathfrak{F}_S(\mathcal{H}_1 \oplus \mathcal{H}_2)$
(proof: Segal and Goodman [1965]).

from which it follows that, explicitly, $L^2(\mathbb{R}, e^{-x^2/2}dx) \otimes \ldots . \otimes L^2(\mathbb{R}, e^{-x^2/2}dx)$ (n times in all) is isomorphic to $e^{\mathbb{C}^n} = \mathbb{C} \oplus \mathbb{C}^n \oplus (\mathbb{C}^n \otimes \mathbb{C}^n)_S \oplus \ldots .$. In particular as $n \longrightarrow \infty$ the Hilbert space of the quantum field, regarded as an inifinite collection of harmonic oscillators, becomes isomorphic to the Fock space over the 1-particle Hilbert space. If the limit is *not* reached we have an isomorphism with the Fock space over a *finite* dimensional Hilbert space. Of course the CCR's cannot be represented in such a space; from the material of Sections 2.4, 3.1 we know that this corresponds to the fact that the spacetime group is non-compact, so that it is only when the collection of oscillators is a literal infinity that we can recover a (reducible) representation of the spacetime group on each n-particle subspace.

We conclude that the meaning of the infinity of degrees of freedom, which in one sense has the same consequences in both field and particle theories (the inequivalence of representations), nevertheless is attatched to different features of the respective mathematical structures that are used: on the one hand, the cardinality of the number of harmonic oscillators determines the dimensionality of the Hilbert space of the states of the particles, whereas the cardinality of the number of particles is associated with (although it does not of course completely determine) the total energy of the collection of oscillators[5].

[4] We remarked on this property of Fock space in (1.3.4), Eq.(28); it also underlies the equivalence of the Hilbert space used in Section 3.3, of the form $\mathfrak{F}(\mathcal{H}_a) \otimes \mathfrak{F}(\mathcal{H}_b)$, with the Hilbert space $\mathfrak{F}(M)$ used in Section 3.4. See fn.1 (3.4.1).

[5] And bearing in mind the point above, even more obliquely requires that the number of oscillators be infinite, in the sense that one could not find any equivalence with a finite set of harmonic oscillators if one begins from a particle theory with an infinite dimensional space of states.

The infinity of the number of harmonic oscillators is of course connected with the fundamental intuitions as to what a classical field is; namely that a classical field admits an arbitrarily chosen set of Cauchy data over at least compact subsets of 3-dimensional space. In fact it is when this Cauchy data is extremely simple, from a set-theoretic point of view, for example when the field is everywhere zero except in a connected regions where it is constant, that the approximation to the field configuration yielded by any finite Fourier series is most unsatisfactory.

In this way it may be that in a certain sense the approach to the limit of infinite degrees of freedom ensures the same thing: from both field and particle viewpoints, that the system is described to an arbitrarily high degree of differentiation, both with respect to the Cauchy data and the subsequent evolution of the system. But if one tries to make more precise this idea of differentiation, one is led to slightly different intuitions. In the particle case this differentiation concerns most specifically the non-compact boosts and translation sub-groups, the fact that they are continuous and extend to infinity; whereas in a field theory, that the Cauchy data for the time-zero field can be specified arbitrarily. This is most easily seen when one considers how to approximate sharp boundaries by a sequence of periodic functions, but in fact however severe the smoothness requirements on a function space one has an infinity of basis vectors (whenever a basis can be defined).

There are various ways in which one would like to generalize this discussion; for example, the interpretation of the field as an infinite collection of harmonic oscillators is obviously too restrictive (and in this respect we differ most emphatically from the recent suggestions of Teller [1989]) and the failure of the superposition prin-ciple for the classical non-linear fields makes it altogether unclear whether we can even begin to formulate the field - particle equivalence along the lines above. We shall not try to say anything very accurate about the non-linear case (*cf.* Segal

[1964], [1967b], [1970]) but a first step is to formulate the state space of the quantum field more directly in terms of field configurations rather than normal modes. In fact, with only a little more sophistication in the use of probability theory on function spaces, one can show a general equivalence of the Fock space with a Hilbert space which is an L^2 space of maps from functionals of the classical field configurations to the reals, that is of the form $L^2(\mathcal{S}'(\mathbb{R}^d), d\mu)$, where $\mathcal{S}'$ is the dual to the Schwarz space of functions on $\mathbb{R}^d$, and where μ is Nelson's measure. The appearance here of the dual to a pre-Hilbert function space is mathematically motivated; the idea first appeared in unpublished work of von Neumann in the mid 1930's, who constructed a Gaussian measure over such a space and showed that it is countably additive on the dual space. The theory first appeared in the literature in Friedrichs [1953], and in a complete form in Segal [1956 a,b].

From the point of view of the geometric quantization this Hilbert space does not appear very natural; when, however, $\mathcal{S}$ is completed in a Hilbertian norm and appropriately complexified $\mathcal{S}$ and $\mathcal{S}'$ become isomorphic and this Hilbert space is canonical in the sense that we have an L^2 space of functions on the classical configuration space of the field. However, more generally, and if one pursues the heuristic offered by the deep connections with the classical theory of stochastic fields, the use of an L^2 space of functions on distributions provides more leverage on the interacting case (but with non-Gaussian measure) and permits states of the field which are much more singular; in the path integral terminology $\mathcal{S}'$ is naturally identified with the "path history" space of a classical stochastic field. In classical probability theory indeed this history space was defined set-theoretically with really only enough structure to carry a σ-algebra of subsets; the gradual realization that one loses essentially nothing by restricting the measure to a much smaller space marked some of the fundamental developments in the subject.

Despite the interest of these ideas our sights are set at a

lower level; we shall content ourselves with the observation that the Hilbert spaces of this form provide the representation in which the field is "diagonalized", at least in the linear case. We may represent this situation in a heuristic way in terms of the Hilbert space of functionals of the classical field configurations, $\mathcal{H} = L^2(Q, d\mu)$, so that for the canonically conjugate quantum fields Ψ, π and with $\psi \in \mathcal{H}$, $\varphi \in Q$:

$(\Psi(\mathbf{x})\psi)(\varphi) = \varphi(\mathbf{x})\Psi(\varphi)$

$(\pi(\mathbf{x})\psi)(\varphi) = -i\hbar\, \delta\psi(\varphi)/\delta(\varphi(\mathbf{x}))$

where the functional differentiation on the RHS of the second of these equations may be defined in a formal sense by

$$\delta\psi(\varphi(\mathbf{y}))/\delta\varphi(\mathbf{x}) = \lim_{\varepsilon \longrightarrow 0} \frac{1}{\varepsilon}\left[\psi(\varphi(\mathbf{y})+\varepsilon\delta^3(\mathbf{x}-\mathbf{y}))-\psi(\varphi(\mathbf{y}))\right]$$

(we shall not attempt to give a precise definition). As emphasised by Isham [1975], Glimm and Jaffe [1972], and many others, this representation is the field equivalent of the Schrödinger representation; the unitary equivalence of the representation thus defined on the Hilbert space $\mathcal{H}$ with the Fock representation of the fields, that exists in the free field case, may be called the mathematical expression of the field - many-particle equivalence. The general idea - that the natural state space for a quantum field is a space of functionals of the field configurations - was explicit in Jordan and Pauli [1928], as we remarked in (1.4.4).

In our view the question of the general validity of this equivalence is the same thing as the question: to what extent do the non-Fock representations (that certainly exist within field theory) have applications in physics? It must be acknowledged, however, that non-Fock representations are frequently interpreted (and indeed constructed) in particle terms, that is as a sort of limit of Fock representations, most specifically as representations in which the number of particles is infinite. As is clear from the foregoing, as long as we proceed from a particle viewpoint, this is the only meaning which can be attatched to non-Fock representations (it is only in this way that one obtains an infinity of degrees of freedom). In most of the applications that we shall consider (in statistical mechanics and the

measurement problem) not only is this heuristic almost universal, but also other physical quantities which are independent of the particle concept (most particularly the total energy) are not defined, and are considered infinite. Therefore there is a very definite sense in which one can insist that such representations are idealizations, from field or particle viewpoints, and that if these idealizations have utility in physical applications, that is only to show that the idealization of a system as consisting of an infinite number of particles is utile in these applications. One concludes nothing about the field - many-particle equivalence.

In the sense of Redhead [1983], [1988], we may interpret the "under-determination" of quantum field theory, with respect to a field or particle ontology, in terms of the physical applicability of non-Fock representations, because a particle interpretation of a quantum field theory is the same thing as defining a Fock representation for that theory. We are now talking about a particle interpretation for a theory which is apparently an idealization; the particle number is infinite, so the representation is non-Fock, but this is precisely wherein the idealization consists. There are so many philosophical questions raised at this point that for the moment we shall only list them:

1. Redhead and others consider primarily perturbation theory in QED. Here too one has an infinity of particles (virtual particles) but this is not normally considered to imply that the theory is an unphysical idealization of the world.
2. There may exist non-Fock representations with physical applications which are not unphysical in any obvious sense (other that they are non-Fock!) and may not be defined as any limit of Fock representations.
3. An idealization is the less innocuous when it can be relaxed without deep modifications of the underlying mathematical model; but the existence of inequivalent representations depends critically on the idealization of infinite particle number, from the particle viewpoint.
4. The non-existence of an operator in a representation does

not necessarily imply that the physical quantity associated with that operator (if it can be so associated) is infinite.
5. On the basis of the philosophy of Section 2.2 (which is partly conventionalist) the idealizations of infinite particle number or total energy may still refer to the world, as mediated by the descriptions which we wish to formulate of the world.
6. Even if there exists a corrupt particle interpretation of a representation, there may exist a field interpretation which is superior, either intuitively or heuristically or both.

Broadly speaking, we shall elaborate on (1) and (2) in (3.5.3) and (3.5.4) respectively, but the remaining questions shall be approached from many different points of view; they constitute the basic theme thereafter.

3.5.3. Haag's theorem.

The excessive reliance of philosophical commentary on perturbation theory in QED is only partly excused by the parallel dominance of perturbation theory in physics over the last half century. It is only in the last decade or so that the dominance of scattering theory has receded, essentially having proved a failure in hadron physics in the middle energy regime. But if the dominant philosophy of practicing physicists is mathematical opportunism, that of philosophers is traditionally inclined to rigour.

Perturbation theory in QED is essentially a collection of rules which allow one to unambiguously extract numerical values, but otherwise devoid of any rigorous mathematical meaning. That remains the case in spite of more than thirty years of intensive effort on the part of a dedicated community of mathematicians, for even the simplest non-linear models. The recent indications that $\lambda\phi^4$ has a trivial S-matrix in four space-time dimensions, and probably QED also (unless embedded in a grand unified theory), leave one in an unprecedented situation in physics, in which one might hope to make mathematical sense of a complete or at

least much more complete theory, but cannot make mathematical sense of a simple model, unless the model is almost trivial.

One rigorous insight into the existence of divergences in perturbation theory, most specifically by the use of the interaction picture, was developed early in the constructive theory of quantum fields and in fact fueled the development of the algebraic approach. The starting point was the analysis of Van Hove [1951] of a wide variety of bose-fermion interactions which showed that neither the total energy H, nor the interaction Hamiltonian H_I, could be defined as operators on the Fock space $\mathcal{H}$ of the free fields, and likewise that for any vector $\psi \in \mathcal{H}$, neither $H\psi$, nor $H_I\psi$, belong to $\mathcal{H}$. By an ironic coincidence, the result of more than two decades of work into perturbation theory led to the first rigorous proof of the inconsistency of the basic method used, in the same year that Dyson established the formal agreement of the Tomonaga and Schwinger renormalization schemes. Penetration into the physics literature took many years; for the most part, the attitude widely shared by the physics community in the early 60's was that Van Hove's arguments were already understood and were adequately dealt with in perturbation theory[6].

More accurately, Van Hove did not quite produce a rigorous argument; that was to come for only one of the interactions which he considered in [1951], namely the model, due to Wentzel [1949 Sec.7] of the Yukawa theory, in which the meson field is treated as a c-number external potential with classical time-independent source distribution $\rho(\mathbf{x})$ ("recoilless nucleons"). In Van Hove [1952] he gave a complete solution to this problem in the physically and mathematically interesting case of a point source; this model and Van Hove's analysis was subsequently studied and improved upon by a number of authors, including Friedrichs

[6] Obviously this response does not really make sense. For further comment see below; more charitably, one might say that most physicists believed the difficulty was purely technical, and was intuitively understood.

[1953 Part III], and Schweber [1961 Sec.12a]; for a clear and simple discussion of this model, and further references, we refer to Emch [1972 Sec.1.1e].

Van Hove also established that if one *assumes* that there exist stationary states of the full Hamiltonian in the more realistic model of two interacting quantum fields, then exactly the same result follows: neither the full nor the interacting Hamiltonian exist as operators on the Hilbert space of the free fields, and either applied to a vector in the free representation yields a vector which is not contained in this representation. This argument was elaborated by Haag and others under more general assumptions and now consitutes what is known as Haag's theorem (Haag [1955], Streater and Wightman [1964], Streit [1968]). There are in fact several related theorems, which can be stated in different ways; we state the theorem using the material of (2.4.9) and Section 2.3, which must be supplemented with a further concept, that of an *invariant mean* over a group. This idea is intuitive enough; if we have a group G which acts automorphically on a C^*-algebra $\mathcal{A}$ as α_g: $\mathcal{A} \longrightarrow \mathcal{A}$, $g \in G$, and $\mathcal{S}$ is the set of states, then $g \longrightarrow \langle \phi; \alpha_g[A] \rangle$ for each $\phi \in \mathcal{S}$, $A \in \mathcal{A}$ is a complex valued, continuous, bounded function on G and one might hope to make sense of the average value of such a function as g ranges over the whole of G. To do so we actually need the second of the two classes of operator algebras determined by groups, as defined by Segal in Segal [1951] (*cf.* (2.5.8)); the set of all such functions on G can be made naturally into a C^*-algebra ($f_1 f_2(g) = f_1(g) f_2(g)$, $f^*(g) = f(g)$, $|f| = \sup_{g \in G} |f(g)|$) with identity ($e(g) = \mathbb{1}$ for all $g \in G$), which we denote A(G), and moreover G has a natural left (right) action by * automorphisms on A(G): $(h.f)(g) = f(hg)$ (respectively $(f.h)(g) = f(gh)$). A state of $\mathcal{A}(G)$ (denote $\mathcal{S}_{\mathcal{A}(G)}$ which is left *and* right invariant is called an **invariant mean**. Every compact, locally compact group, and every abelian locally compact group, admits an invariant mean (in particular the Euclidean group); no semisimple, non-compact, locally compact group admits an invariant mean (therefore not the Lorentz group, homogeneous or inhomogeneous).

Consider now an (abstract) Weyl system $(\mathcal{W},\mathfrak{S})$ over a symplectic manifold M; a Fock representation is assured if there exists a state ϕ in $\mathfrak{S}$ with the properties of a vacuum (we then define the Fock representation by the GNS representation generated by ϕ). So far we have defined the vacuum state by means of properties abstracted from the Fock-Cook representation, and then found to be sufficient properties: for example, the invariance under the space-time group, the requirement that the generator of time evolutions has positive spectrum on $\mathcal{H}_\phi$, and the existence of a number operator which has positive spectrum on $\mathcal{H}_\phi$; or the invariance of the state under all "one-particle" unitary transformations, when the representation is derived from a positive compatible complex structure, together with positivity of the energy. In formulating Haag's theorem, we want the weakest possible characterization of the vacuum. Following Streit [1968] and Emch [1972] we say that an arbritrary state $\phi \in \mathfrak{S}$ is **η-clustering** with respect to an invariant mean $\eta \in \mathfrak{S}_{\mathcal{A}(G)}$ if

$\eta\langle\phi;\ \alpha_g[A]B\rangle = \eta\langle\phi;\ \alpha_g[A]\rangle\langle\phi;B\rangle$ for all $A,B \in \mathcal{W}$

(note that α is here the assumed action of G on $\mathcal{W}$, not on $\mathcal{A}(G)$, and that we regard $\langle\phi;\alpha_g[A]\rangle$ etc. as a function of g, say f(g); then $\eta\langle\phi;\alpha_g[A]\rangle$ is more propertly written $\langle\eta;f\rangle$, that is the expectation value of $f \in \mathcal{A}(G)$ with respect to $\eta \in \mathfrak{S}_{\mathcal{A}(G)}$).

This assumption is weaker but closely associated with the clustering properties of the Fock-Cook vacuum and holds in all the standard theories; for $\phi \in \mathfrak{S}$ which is also G-invariant (so that $\phi \in \mathfrak{S}_G$ in the notation of (2.5.2)) ϕ is η-clustering if

$\eta\langle\phi;\ \alpha_g[A]B\rangle = \langle\phi;A\rangle\langle\phi;B\rangle$

that is, if the vacuum expectation values of the product of two local fields decouples when averaged over all G-transformations of one of the fields. When G is the spacial translation group we have a reasonably clear *a priori* basis for this assumption. We may now state Haag's theorem, when G is this group:

Theorem 3.5.2

Let $\pi(W)$ be a representation of the Weyl system W, $\mathfrak{S}$ over a symplectic manifold M, with cyclic vector $\phi \in \mathcal{H}_\pi$; suppose that the state $f_\phi \in \mathfrak{S}$ corresponding to ϕ is G-invariant and η-clustering, and that there exists a Fock representation π' of W with vacuum Ω, which also exists as a vector in $\mathcal{H}_\pi$; then if M is infinite dimensional $\Omega = \lambda\phi$ where $|\lambda| = 1$.
(for proof see *e.g.* Emch [1972 Th. III.1.8]; the main burden of the proof is to show that ϕ is the only G-invariant vector state in $\mathcal{H}_\pi$. The fact that f_Ω is also G-invariant is straightfoward.)

This theorem clearly prohibits the physical vacuum (which is supposed to be G-invariant and η-clustering) and the "bare" vacuum (the no-particle state, which is also G-invariant and η-clustering but which is defined with respect to some Fock representation as the no-particle state) from existing as distinct vector states in a Hilbert space on which ϕ is cyclic. Vacuum polarization cannot occur.

There is also a theorem which generalizes Van Hove's observation that the interaction Hamiltonian cannot exist as an operator on the same representation space as the free Hamiltonian. This needs the slightly stronger condition that G is η-*abelian*, and we refer to the literature for details of this concept and the precise formulation of this theorem. A heuristic formulation is as follows: if at any time the interaction Hamiltonian is unitarily equivalent to the free Hamiltonian, then it is unitarily equivalent to the free Hamiltonian at all times. Consequently the S-matrix is unitarily equivalent to the identity at all times.

Haag's theorem does not mean that an interacting field cannot be asymptotically free; however it prohibits the use of such artifices as the Moller matrices or the U matrices[7],

[7] As used in, for example, Bjorken and Drell [1965 Sec.17.2]; these authors confuse Haag's theorem with the fact that the proof for the existence of the U matrices fails for systems of infinitely many degrees of freedom (not only does the proof fail: the U matrices cannot exist). Their "solution",

the Tamm-Dancof scattering theory and Dyson's use of the interaction picture. As a result the entire edifice of traditional perturbation theory fails, and there is no doubt that some of the divergences characteristic of the theory stem from these difficulties (*cf.* Emch [1972 p.27]). Streater has remarked, that Haag's theorem "is one of the most widely misunderstood results of the subject" (Streater [1975 p.796]). By the early 60's, at any rate, the significance of this theorem was not properly understood, not even by the experts. We refer to the exchanges between such notables as Gunnar Källen, Werner Heisenberg, Arthur Wightman, Leon Van Hove, and Y. Nambu (and in the presence of Eugene Wigner and Paul Dirac amongst others), on the proper interpretation of Haag's theorem at the 1961 Solvay lectures:

> W. Heisenberg: In connection with the so-called "theorem of Haag" I would like to point out, that its content should be well known already from conventional quantum mechanics. If one compares, *e.g.* the state of a ferromagnet where the total magnetic moment has the direction of the z-axis and another state with a slightly different direction of the total mangetic moment, these two states will always be completely orthogonal to each other, if we have to do with an infinite ferromagnet...therefore...one must be careful not to write down relations between matrix elements in which both sides of the equation are trivially zero....such an error might occur in perturbation theory or in the old Tamm-Dancoff method, where one starts with the "bare" vacuum; it ought however not to occur in the new Tamm-Dancoff method, where one starts from the real vacuum...
> G. Källen: I agree very much that the theorem discovered by Van Hove and Friedrichs and usually referred to as the "Haag theorem" is really of a very trivial nature and it does not mean that the eigenvalues of a Hamiltonian never exist or anything that fundamental...however I do think that this theorem does show that the old fashioned Tamm-Dancoff method is essentially not better than perturbation theory and I do not believe that the new Tamm-Dancoff method is so much better. The fundamental difficulty is that a finite amount of probability (one) has to be divided between so many states that each state gets essentially zero probability. This problem remains also in the new Tamm-Dancoff method.
> A. Wightman: ...the significance which one attaches to Haag's theorem depends on one's attitude towards a model such as Heitler's. On the one hand, one can regard this model as a short hand for the investigation of the numerical effect of cut-offs in the perturbation series of

to postulate the existence of U(t), is absurd.

a relativistic theory. Then mathematical questions about the exact spectrum of the model are quite irrelevant. On the other hand, one can take the Hamiltonian of the model really seriously....in this case, it seems to me that Haag's theorem is distinctly non-trivial. It says that to make physical sense of the Hamiltonian one must insert not the familiar representation of the annihilation and creation operators but one of the strange representations.
L.Van Hove: I would like to make a few remarks on the question of the expansion (of the interacting states in terms of the free states)...the formal difficulties connected with this expansion originate from the fact that all (the matrix elements) become zero in a realistic situation. This can be due to two completely different causes which should be sharply distinguished.
In the case of an interaction modifying the physical system over the whole of space...(the matrix elements) are zero because of the infinite extension of space: this is seen by enclosing the system in a finite volume V, calculating (the matrix elements) for V finite and noticing that for $V \longrightarrow \infty$ (they) go to zero...this situation holds even in a field theory with cut off, we know how to handle it and it is not connected with the real difficulties of field theory (nevertheless Haag's theorem, if I understand it correctly, refers to this situation and is therefore, I think, of little direct relevance to the basic difficulties of field theory).
The second case where one knows that (all the matrix elements tend to zero) in the case of point particles interacting with a quantized field, the interaction acts in a limited region of space only.....
Y.Nambu: ....but perhaps we should keep in mind the possibility that Haag's phenomenon can arise from two different physical reasons, namely the continuous nature of space time and the practically infinite volume of the universe. (Stoops [1962 p.169-173]).

There can be no question of resolving the questions raised by this exchange in this thesis. However, we remark that Heisenberg's comment is wholly inappropriate, insofar as the "conventional quantum mechanics" he appeals to is already a non-Fock representation, and that with respect to Van Hove's remark, even if we do know how to deal with the infra-red divergences in formal terms this does not mean we do not learn something important about the appropriate rigorous mathematical formulation of the problem.

What is now generally agreed (*cf.* Streater [1975]) is that there are a variety of methods which can be used to circumvent the difficulties of Haag's theorem in the scattering problem (but not where there are bound states) which do not explicitly address the problem of determining the physically correct representation (the Haag scattering

theory or the LSZ theory in particular). In these theories whilst one can define an S-matrix the evolution cannot be represented unitarily in the Fock representation; on the other hand, it may still be defined as a continuous group of automorphisms of the algebra (Guenin [1966], Segal [1967a], Glimm and Jaffe [1972]). Further, it is still possible to represent the physical field and its asymptotic limits as $t \longrightarrow \pm \infty$ (in which the field becomes free) on the same Hilbert space, although they are not related by any unitary transformation. Finally, the assumption of asymptotic completeness is not directly prohibited by Haag's theorem (*i.e.* that $\mathcal{H} \simeq \mathcal{H}^{\pm}$ where $\mathcal{H}$ is the Hilbert space of the interacting fields, and $\mathcal{H}^{\pm}$ the asymptotically free limiting states), so that the interacting states can consistently be regarded as unitarily related to the asymptotic states, or equivalently every state in $\mathcal{H}$ can be interpreted as a superposition of asymptotically free states.

However, this possibility does not vitiate the virtual particle interpretation of the interacting field derived from the traditional perturbation theory; it only means we can define the state of the interacting field in terms of a superposition of mass-shell (*i.e.* real, not virtual) particles (precisely that superposition which eventually evolves out of, or into, the interacting state). The virtual particle interpretation is one formal aspect of perturbation theory which is definitely undermined by Haag's theorem; quite apart from metaphysical questions as to the validity of the particle concept, when superpositions of states of different numbers of particles are permitted (Weingard [1988], Teller [1988])[8], the reality of virtual particles

[8] Obviously I do not share their qualms; superpositions of this sort are possible in any quantum field theory, including the free theory, in the Fock representation, which I have consistently regarded as the bedrock of what it means to give a particle interpretation of a quantum field. The ontological status of a superposition of states of different particle number is precisely, no more and no less, equivalent to the ontological status of superpositions of states of different energies. One does not conlude that the concept of energy has no ontological reference for this reason.

rests on a demonstrably inconsistent formalism.

One final point: the strange representations to which Wightman refers, and which are usually demanded if we are to define the Hilbert space of the interacting theory, are not non-Fock representations. As we indicated in (2.5.9) a Fock representation is uniquely defined by the generating functional

$$\phi_F(f) = e^{-|f|^2/2}$$

where $f \in M_J$. Only the imaginary part of the norm on M_J is determined by the symplectic form on M; the real part (which enters into the exponent $|f|^2$) depends on the choice of complex structure on M. There are an infinity of inequivalent Fock representations, with different real parts to the inner product $<.,.>_J$ that we must define on M. Haag's theorem but indicates the need, which we have already referred to (3.4.11), of "fine-tuning" the representation to the interaction considered. We now turn to non-Fock representations proper.

3.5.4. The Fulling pathology.

The problems of quantizing a classical mechanical theory in an arbitrary co-ordinate system are well known; the geometric quantization was developed partly to eliminate the formal dependence of the known quantization schemes on Cartesian co-ordinates. These difficulties are even more accute in quantum field theory; the discovery, due to Stephen Fulling, that a "naïve" application of the quantization of the free scalar field in 2 spacetime dimensions (expansion into normal modes, and then replacing the amplitudes by creation and annihilation operators) leads to a completely different particle interpretation for the theory when Rindler co-ordinates are used, initiated a revival of interest in this problem. In the following years a number of fundamental insights were obtained on some old problems, such as the dependence of the particle concept on a global spacetime symmetry group, and the Casimir effect, as well as altogether new ideas such as Hawking radiation and the observer dependence of the particle content of a

fixed region of spacetime.

The connection between these ideas goes something as follows. Rindler co-ordinates are defined on only part of (two dimensional) Minkowski space (fig. (3.5.1)), as is obvious from their definition: Rindler coordinates z,ν, $0 < z < \infty$, $\nu \in \mathbb{R}$ are defined by:

$t = z \sinh \nu$, $x = z \cosh \nu$.

Correspondingly the quantization in these co-ordinates is essentially a quantization on a manifold with boundary. The conventional formulation of this problem is, of course, to impose boundary conditions on the states, and it was in this way that Cassimir was led to evaluate the energy density of

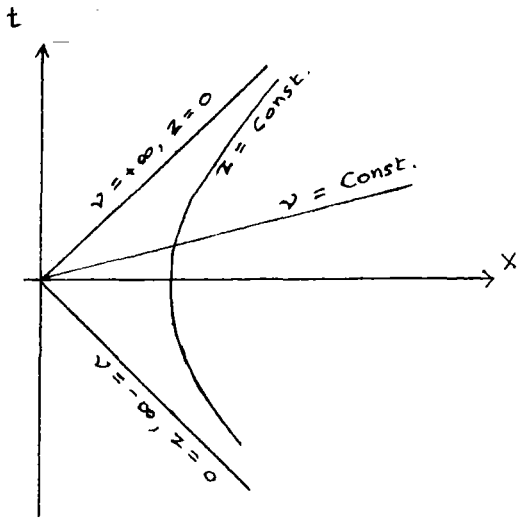


Fig. 3.5.1

an electromagnetic field in its ground state between two infinite conducting sheets (Casimir [1948]). The (experimentally confirmed) force on the conductors which is theoretically predicted is in some sense evidence that the ground state differs from the Fock vacuum. This suspicion is confirmed by Fulling's discovery, that the vacuum state defined as the no-particle state in Rindler co-ordinates differs from the usual Fock vacuum, and in fact contains a thermal distribution of particles with respect to the usual Fock vacuum.

In his first publication on this effect Fulling ([1973])

entitled his paper "Nonuniqueness of Canonical Field Quantization in Riemannian Space-Time", reflecting his belief that the obvious physical desideratum picking out the "true" vacuum - that is, whether there does, or does not, exist a *physical* boundary the presence of which leads to unambiguous observable effects (as in the Casimir effect) - is not in general available for physically more realistic spacetimes, in which the metric is not stationary (*i.e.* does not admit an everywhere timelike Killing vector field). In Fulling's masterfull account of the general situation most of the basic themes which have subsequently been explored were spelled out:

Completeness of the spacetime manifold - if there are no *physical* boundaries to spacetime, a quantization scheme must always be defined on all of spacetime - in the Schwarzchild case, for example, this means that the usual co-ordinates (with a fictitious singularity at the Schwarzchild radius) are unsuitable for a quantization scheme, since they would correspond to a physical barrier (perfectly reflecting sphere) at the Swarzchild radius (Duetsch and Candelas [1979]).

The particle concept; local properties are defined by global considerations - evidently the particle interpretation of a quantum field is determined by topological features of the manifold; for another example, quantization in a box with periodic boundary conditions defines a different vacuum from the usual one.

Distinguished co-ordinate systems - different ways of slicing a manifold into "space" and "time" result in different quantizations (in Wheeler terminology: physical intuition tells us that the universe is a vast haystack of particle paths, but the theoretical apparatus presently available to us forces us to treat the universe as a stack of automobile fenders.)

Possible superiority of local (field) observables - even in favorable cases (static and stationary spacetimes) the unambiguous objects in the theory may be local functions of the fields; one must dispense with the particle interpretation of a theory.

To these ideas we may add:

Observer-dependence of the particle concept - any given Lorentz frame is "natural" for an observer at rest in that frame (kinematic interpretation of symmetry, *cf.* (2.5.2)); correspondingly at least some non-inertial co-ordinate systems are natural for non-geodesic observers (dynamical interpretation of symmetry), and the use of such co-ordinates in a quantization scheme may yield a particle interpretation which corresponds to the objective particle phenomena apparent to such an observer (the Unruh effect, Unruh [1974], Sciama, Candelas and Deutsch [1981], Birrel and Davies [1982], Davies [1984]).

Quantization on tangent spaces - in view of these difficulties, and to circumvent the apparent dependence of a particle interpretation on global properties of the spacetime manifold, one should perhaps define a particle interpretation on the tangent space at points of spacetime; each tangent space to a (pseudo-Riemannian) manifold is Lorentzian, and one may attempt to associate the inhomogeneous Lorentz group to the tangent space so that the translations correspond to infinitesimal displacements in the tangent space (Martellini [1982]).

Each one of these ideas has generated controversy; what little concensus there is centres on the fourth, as a positive strategy for those field observables which do not depend on normal ordering for their definition (obviously normal ordering depends on the correct definition of the creation and annihilation operators, and therefore on the particle interpretation of the theory). See, for example, Fulling [1983], [1984]. In addition phenomena such as Hawking radiation , which can be described in terms of the difference between the vacuum defined by Schwarzchild co-ordinates, and the Fock vacuum defined by Cartesian co-ordinates, is held with a measure of confidence by cosomologists not least because of its elegant formulation in terms of classical thermodynamics (Bekenstein [1973], [1983]), and Hawking's original analysis of the process of stellar collapse (Hawking [1974]).

In this situation, and in the limited space available, we shall only describe the connection with complex structures as formulated in the previous section. As commented by Fulling ([1973]) the canonical quantization on a manifold with static metric (where one has a global Killing vector field) makes critical use of the Killing vector in defining creation and annihilation operators (as positive and negative frequency parts of the field). According to the theory of Section 3.4, however, the creation and annihilation operators are defined by the complex structure J on the classical solution manifold via Eq.(3)(3.4.2):

$$a(\phi) = \frac{1}{\sqrt{2}} (A(\phi) + iA(J\phi))$$

$$a^*(\phi) = \frac{1}{\sqrt{2}} (A(\phi) - iA(J\phi))$$

(we shall consider only the real scalar field). For the *particle* complex structure, the two methods amount to the same thing; the definition of J depends on the unambiguous decomposition of classical field solutions into positive and negative frequency solutions.

The usual (Minkowski space) method of defining decomposition - that is by use of the Fourier transform - can be generalized to just this: a *static* spacetime, in which there exists a global timelike Killing vector field which is everywhere orthogonal to a family of spacelike hypersurfaces. The dependence of the decomposition on *global* properties of the spacetime (the non-local dependence of $\varphi^{\pm}$ on φ is only one aspect of this) makes clear the sense in which the particle interpretation (and thereby the particle concept) is partly determined by the structure of the spacetime at infinity. However counter-intuitive[9] the idea, from a field point of view the particle content on a spacelike hypersurface cannot be defined by information pertaining solely to that hypersurface[10].

[9] On the other hand on the philosophy of Section 3.1, we do not really know how to define the particle concept in the absence of a global symmetry group, which leads to a similar conclusion.

[10] One might wonder what a more precise definition of this statement would amount to - would it apply if one formulates

We recall (3.4.3) that we suppose M (as a real vector space) comes equipped with a symplectic form ω which defines the imaginary part of the Hermitian inner product $<.,.>_J$ on M_J (independent of the complex structure J). The choice of J determines the real part of $<.,.>_J$. One circumstance in which the real part of $<\varphi,\varphi'>_J$, $\varphi,\varphi' \in M \subseteq M_J$ has obvious physical significance is when $\varphi' = H\varphi$ (it is then the expectation value of the energy); H also depends on J (H is that operator on M_J which generates the unitary transformation on M_J corresponding to the evolution on M), specifically H determines the Schrödinger equation (Eq.(35)(3.4.4):

$$H\varphi = J(\mathcal{L}_t\varphi)$$

(we have replaced the partial time derivative of φ by its Lie derivative along the time-like vector field defined by the Killing vector field on spacetime). Therefore we obtain:

$$<\varphi, H\varphi>_J = \omega(J\varphi, H\varphi) - i\omega(\varphi, H\varphi) = \omega(\varphi, \mathcal{L}_t\varphi) - i\omega(\varphi, J\mathcal{L}_t\varphi).$$

For the real field the second term on the RHS is always pure imaginary; since the first term on the RHS is independent of J we see that the positivity of the energy, which distinguished the particle from the natural complex structure for the complex scalar field, has here no bearing on the choice of J. In fact the requirement that the expectation value of the energy be real is the only constraint, and demands that:

$$\omega(\varphi, J\mathcal{L}_t\varphi) = 0$$

It is a theorem (Ashtekar and Magnon [1975]) that this constraint uniquely determines the complex structure J for an arbitrary static spacetime. In fact, these authors were able to show that this is true even for *stationary* spacetimes, that is, where there exists an everywhere timelike Killing vector field (but where it is not necessarily orthogonal to a family of spacelike hypersurfaces).

the properties of spacetime in position space for example? One suspects that to define any generalization of the notion of position space to non-Minkowski spacetimes one would first need to solve the problem as posed.

Ashtekar and Magnon formulated this theorem in terms of one expression of the field - many-particle equivalence. In the quantization of Section 3.4 for linear fields, a classical field solution is at the same time a (not necessarily normalized) element of the one-particle subspace of the Fock space and further, the energy of the field in this configuration is equal to the energy of this one-particle state: *i.e.* one has:

$$\langle \varphi, H\varphi \rangle_J = \int T_{ab} t^a dS^b \qquad (2)$$

where T is the stress-energy tensor, and t^a a unit time-like vector; the integral is over a spacelike Cauchy surface. For the real scalar field the *postulate* that Eq.(2) holds imposes no further constraint on J than that H is hermitian (*i.e.* the constraint on the real part, that $\omega(\varphi, \mathcal{L}_t \varphi) = \int T_{ab} t^a dS^b$, is automatically satisfied in this case). This is no longer true for the complex scalar field (although it is equivalent to the condition that the expectation value of the Hamiltonian is positive definite). However, and unlike the physically transparent condition that the Hamiltonian be e.s.a. and positive, this condition makes no sense in the non-linear case, where it can only be formulated in an asymptotic way (and this is fraught with difficulties in non-trivial spacetimes).

We conclude that the moral of the study of free field theory on curved spacetimes is that a particle interpretation is unambiguous only in special circumstances, most particularly when the metric is stationary, or (optimistically) when this condition holds asymptotically so that the particle flux at infinity can be defined; the many fascinating and philosophically fundamental questions that are raised, most particularly the possible observer-dependence of particle phenomenology, require a much more comprehensive treatment. In these circumstances we shall not attempt a superficial commentary. We refer instead to the literature: apart from the citations above, see Isham [1975], Hawking [1975], Unruh [1976], Davies and Fulling [1977a,b], Gibbons [1979], Lapedes [1978], de Witt [1979], Deutsch and Najmi [1983], and Hinton [1983].

3.5.5. Thermodynamics and statistical mechanics[11].

The most developed theory of non-Fock representations is to be found in the study and definition of the thermodynamic limit, under the (usually explicit) proviso that the representations constructed for this purpose are non-physical in this central respect: the size, mass, and particle number of such models are all infinite. We shall first briefly describe some of the basic ideas of these applications; in the following sections we shall consider their interpretation more deeply.

The basic ideas are well illustrated in the Araki and Woods [1963] model of the infinite bose gas, one of the first applications of non-Fock representations to physics. These authors sought an infinite-volume finite density model; the first point is that obviously the density of a gas must be zero in any Fock representation for infinite volume, because Fock space contains only finite-particle arbitrary states, and the number of particles for finite density must be infinite. Of course, the time-honoured strategy is to calculate everything that one wants to know in a finite volume, and then let this volume tend to infinity (and the number of particles also) whilst keeping the density constant, to obtain the limiting thermodynamic behaviour of the system. The key insight of Araki and Woods was that the density operator, if it exists, must be a constant in any given representation. In an irreducible representation it is easy to see this from a straightfoward calculation of the infinite volume limit of the commutator of a bounded function of the density operator with an arbitrary element $W(\varphi)$ of the Weyl algebra; this commuator vanishes in the limit, so that the density operator must lie in the commutant of the Weyl algebra. In particular, in an irreducible representation the commutant is trivial and the density operator must then be a multiple of the identity. In the Fock case, since this representation is irreducible, its

[11] With minor modifications the rest of the material in this section is contained in Saunders [1988].

value can be calculated on any state, in particular the vacuum, where of course its value is zero as expected.

It actually follows that the density must be a constant on any cyclic representation, with a value given by its expectation value on the cyclic vector. We now note that although the Fock representation does not contain arbitrary infinite particle states we can still use this representation to define states of this character; one defines a finite density functional $\phi_{F,V}$ on M_V (the classical solution manifold satisfying periodic boundary conditions on the boundary of the volume V), which generates a finite-volume Fock representation, and then lets V tend to infinity. Both $\phi_{F,V}$ and its infinite volume limit ϕ_F are states in $\mathfrak{S}$; the latter defines a new representation by the GNS construction, which, as Araki and Woods showed, describes a system of infinite volume and finite density. They also found that the density operator always lies in the centre of the resulting representations. This being so, it may be considered a classical observable, in the sense of (2.3.5).

This strategy for defining non-Fock representations for the infinite-volume limit is quite general; one must start with the finite volume system in the Fock representation, pick a state which is in some way typical of any finite sub-volume of the infinite system one wants to describe, allow the volume to go to infinity to arrive at a new state (which will not lie in the original Hilbert space), and then use this state to define a new representation. We shall speak of such representations as *thermodynamic* or *collective* representations. In general the states which one arrives at in this way are not pure, so that one ends up with reducible representations. One might expect this to be so, because application of elements of the algebra to vectors in the Hilbert space of the Fock representation result in a change in particle number; on this intuition, and since an infinite system of non-zero density cannot be affected by any finite change in particle number, there should be some degeneracy

present to reflect this fact[12].

Araki and Woods also extended their analysis to the finite temperature case with and without macroscopic occupation of the ground state; the general feature seems to be that intensive properties such as entropy density, chemical potential, temperature, and density are uniquely associated with a representation. Significant results have also been obtained in connection with the Bardeen, Cooper, and Schrieffer model of superconductivity (BCS model; see Schrieffer [1964], Haag [1962], and Jelinek [1968]) and for a class of Weiss models for ferro- and antiferromagnetism (Emch and Knopps [1970]). In all these cases the states which generate these equilibrium representations satisfy the *KMS condition* (after Kubo [1957] and Martin and Schwinger [1959]), first formulated as a boundary condition on the analytic behaviour of thermal Green's functions.

This condition arises naturally as an equilibrium condition, as can be seen from the following argument: consider the Gibbs state for a finite volume system $\rho = e^{-\beta H}/Tr[e^{-\beta H}]$ at a temperature $\beta=1/kT$ with Hamiltonian H; under the automorphic time evolution α_t one has:

$$\langle \rho;\ \alpha_t[A]B\rangle = \frac{Tr[e^{-\beta H} e^{iHt} A\, e^{-iHt} B]}{Tr[e^{-\beta H}]}$$

$$= \frac{Tr[e^{-\beta H} B\, e^{i(t+i\beta)H} A\, e^{-i(t+i\beta)H}]}{Tr[e^{-\beta H}]} = \langle \rho;\ B\alpha_{t+i\beta}[A]\rangle$$

In particular for B=1 and analytically continuing to imaginary time, one sees that ρ must be time-invariant. Quite generally, any state ρ statisfying $\langle \rho;\ \alpha_t[A]B\rangle = \langle \rho; B\alpha_{t+i\beta}[A]\rangle$ is called a **KMS state** for the temperature β; this condition is called the **KMS condition**. It can be shown that a Gibbs ensemble of a finite system with temperature β and chemical potential μ for a finite system satisfies the KMS condition as the volume tends to infinity (with μ,β fixed) if the infinite volume limit of the local densities

[12] Note that the particle number is not defined in non-Fock representations.

exists. The group of automorphisms with respect to which the generating state is KMS may be identified with the abelian Lie group of time translations and gauge transformations (one for each species of particle); see Haag *et al* [1967]. On the other hand, any time-invariant state which is stable with respect to local perturbations is KMS with respect to such a symmetry group (Haag and Trych-Pohlmeyer [1974]). The one exception is when the state is a ground state. This is a limiting case of a KMS state, and remarkably (Kastler and Takesaki [1979]) the KMS condition there goes over to the positivity of the spectrum of the Hamiltonian (and the limiting KMS state may be identified with the vacuum).

The KMS states are the most important state-based characterization of the *classical properties* of a system; such observables may also be characterized directly, and we conclude with two methods for doing this. The first explicitly emphasises the intuitive notion of a classical observable as the mean value of some microsopically defined observable. There are many ways of doing this, but a general formulation may be defined when a symmetry group G is present: consider again an invariant mean η (as defined in (3.5.3)); η defines a natural mapping η: $A \longrightarrow A_\eta$ from $\mathcal{A}$ to its double dual $\mathcal{A}^{**}$ by $\eta\langle\phi;\ \alpha_g[A]\rangle = \langle\phi;\ A_\eta\rangle$ such that $\alpha_g[A_\eta] = A_\eta$ (here α is the extension of α to $\mathcal{A}^{**}$). For each $A \in \mathcal{A}$, we may think of A_η as the mean of A over G. We shall call such observables **global observables**. That a global observable is classical follows if we can show that A_η commutes with all elements of $\mathcal{A}$. This is plausible when G is the group of space-translations; the fact that $\mathcal{A}$ is the algebra of local observables together with microcausality appear intuitively to demand commutativity. However, this property does not follow merely from the definition of A_η: in fact, it cannot even be formulated as it stands, because A_η will not in general exist in $\mathcal{A}$. Instead one must go to the univeral representation π_u of $\mathcal{A}$ as operators in $\mathcal{H}_u$. One may then *postulate* that η is a map from $\pi_u(\mathcal{A}) \longrightarrow \pi_u(\mathcal{A})' \cap \pi_u(\mathcal{A})''$. A closely related condition, which avoids the introduction of the univeral representation, is that for all $\phi \in \mathcal{G}$, and all $A,B \in \mathcal{A}$, one has:

$\eta<\phi;\alpha_g[A]B-B\alpha_g[A]> = 0$

When this condition holds the system $(\mathcal{A},\mathfrak{G},G,\eta)$ is said to be **η-asymptotically abelian** (Doplicher **et** *al* [1969]); for further discussion, and several related conditions, see Emch [1972 2.2d]).

The other formulation of the idea of a classical observable that we shall consider is due to Lanford and Ruelle [1969]. Consider again the construction of the quasi-local algebra (2.5.3); for each open bounded region B in spacetime (denote the set of all such $\mathcal{B}$) let $\mathcal{A}(B)$ be the quasi-local algebra generated by all A(D), $D\in\mathcal{B}$ with $B\cap D = \varnothing$. For any representation π of $\mathcal{A}$ the von Neumann algebra $\mathcal{L}_\pi = \bigcap_{\mathcal{B}} \pi(\mathcal{A}(B))''$ is called the **algebra of observables at infinity in the representation** π. It automatically follows that $\mathcal{L}_\pi \subseteq \pi(\mathcal{A})' \cap \pi(\mathcal{A})''$ so that an observable at infinity is classical.

3.5.6. Theory of measurement: general theory.

In what follows we shall frequently speak of *macroscopic observables*; these are observables, in the widest sense of the term (*i.e.* not necessarily self adjoint elements in a C^*-algebra), which we suppose figure directly in our experience of the world, and which therefore *ought* to be classically described. We suppose that the basic problem of measurement theory is to show that such observables are indeed classical; from the preceding remarks we see that at least some macroscopic observables (the density of an infinite gas, for example), are classical, because they appear as c-numbers associated with thermodynamic or collective representations. This fact opens the way to an approach to measurement theory on the basic philosophy that a measurement process always arises as a collective or thermodynamic phenomenon (and, more generally, that classical or directly observable phenomena always have this character). The simplest and most telling argument here is that such processes are always irreversible.

This emphasis, on the irreversible character of measurement,

is equally a feature of the Copenhagen interpretation and its subjectivist elaborations; what distinguishes the present approach is that one is able to show that properties defined in the thermodynamic limit are truly classical in the probabilistic behaviour as a purely mathematical consequence of quantum theory.

The measurement problem is, however, specifically a realist issue. It will not be solved until one provides an analysis free of any critical dependence on unphysical idealizations. In the present context that means relaxing the thermodynamic limit: what then emerges is that there is a precise parallel between thermodynamic behaviour and classical behaviour; both must be understood as idealizations.

The first task, however, is to understand the positive contribution of the present approach to the measurement problem. It will be important in what follows to distinguish the states in $\mathfrak{S}$ which are given as vector states $\psi \in \mathcal{H}_\pi$ in a representation π; any such ψ defines a state in $\mathfrak{S}$ which we shall denote $j_\pi(\psi)$ (where there is no ambiguity we may denote this state $j(\psi)$). The central fact to be exploited is that not all the states in $\mathfrak{S}$ arise in this way (nor as convex sums of the form $a_1 j(\psi_1)+a_2 j(\psi_2)+ \ldots$ with $a \in \mathbb{R}$, $\Sigma_i a_i = 1$, which correspond to the density matrices on $\mathcal{H}_\pi$). The states in $\mathfrak{S}$ which *can* be obtained in this way we denote $\mathfrak{S}_{j(\pi)}$; another set of states which we shall need is the set of all states on $\pi(\mathcal{A})$, which is itself a (concrete) C^*-algebra. We denote this set $\mathfrak{S}_\pi$.

Recall that the essential difficulty in measurement theory is to describe the transition $j(\lambda_+\psi_+^M + \lambda_-\psi_-^M) \longrightarrow |\lambda_+|^2 j(\psi_+^M) + |\lambda_-|^2 j(\psi_-^M)$, $\lambda_\pm \in \mathbb{C}$, $|\lambda_+|^2+|\lambda_-|^2 = 1$, that must surely be effected in a simple two-valued measurement M with final state $j(\psi_\pm^M)$ corresponding to the possible experimental outcomes. The fundamental problem is most succinctly expressed in the Schrödinger cat paradox. Classical properties are invariably subject to an ignorance interpretation of probability; that is to say, it must make sense to assume, at any instant, that they have some

definite (if unknown) value. This interpretation of coherent superpositions is known to be untenable; from the analysis (Section 2.2) of propositional systems which admit coherent superpositions, we know that their lattice of propositions must be non-Boolean. We have therefore arrived at the classic measurement problem: how to effect the transition from coherent superpositions to incoherent mixtures. A less widely appreciated problem also arises: how to show that the resulting mixtures (convex sums of states) determine a unique classical probability theory over the experimental outcomes (see below).

The expectation value of some observable $A \in \mathcal{A}^M$ (the algebra of observables associated with the system plus measurement apparatus) in the coherent superposition $j(\lambda_+\psi_+^M + \lambda_-\psi_-^M)$ differs from that in the incoherent mixture by cross terms of the form $|(\lambda_+\psi_+^M, A\lambda_-\psi_-^M)|^2$, where $(.,.)$ is the inner product on the representation space $\mathcal{H}$. This leads to the idea that if all such cross-terms vanish for the *macroscopic* observables of a system, then the foregoing difficulties do not arise, because this description will then be equivalent to the description of the system as in an incoherent mixture, with respect to all macroscopic observables.

This idea plays a central rôle in the Daneri, Prosperi, and Loinger (DLP) theory of measurement (Daneri *et al* [1962]), where the macroscopic observables are those which do not induce transitions between states belonging to distinct "channels" of the measurement apparatus. These observables are typically time-averaged (Heisenberg picture) observables. A similar idea underlies Jauch's ([1964]) use of the partial trace to reduce pure states (of the composite system) to mixtures (with respect to the macroscopic - actually all - observables of the measuring instrument).

In the present theory one can go much further, with no appeal to time-averages or conditionalization over the states of the microscopic system. It is a remarkable fact that in the algebraic approach one can actually show that not only do cross sterms vanish for a large class of

macroscopic observables, but for the whole algebra of observables. To understand this result, we need the notion of *disjointness.* Consider any faithful representation π of a C^*-algebra $\mathcal{A}$, so that vectors ϕ_1, ϕ_2 in $\mathcal{H}_\pi$ determine pure states $j(\phi_1)$, $j(\phi_2)$ in $\mathcal{S}$ and their associated GNS representations π_1, π_2 are irreducible. Obviously each of these representations is unitarily equivalent to some sub-representation of π (the restriction of π to some stable subspace in $\mathcal{H}_\pi$, namely that on which ϕ_1, ϕ_2 respectively are cyclic. Will they be unitarily equivalent to each other? The answer is affirmative if for some $A \in \mathcal{A}$, $(\phi_1, \pi(A)\phi_2)_{\mathcal{H}_\pi} \neq 0$. Any two states $w_1, w_2 \in \mathcal{S}$ are said to be **disjoint** when no sub-representation of π_{w_1} is unitarily equivalent to *any* sub-representation of π_{w_2}; it follows easily that states w_1, w_2 are disjoint if and only if for every representation π such that there exist ψ_1, $\psi_2 \in \mathcal{H}_\pi$ with $w_i = j(\psi_i)$, $i = 1,2$, and for every $A \in \mathcal{A}$ one has $(\psi_1, \pi(A)\psi_2)_{\mathcal{H}_\pi} = 0$. Note also that no finite-dimensional Weyl system possesses disjoint states.

As we have seen ((3.5.5)) macroscopically distinguishable *irreducible* representations are unitarily inequivalent, hence their states are disjoint. The fact that such states differ in the expectation values of macroscopic observables alone is enough to ensure that *all* cross-terms vanish for every $A \in \mathcal{A}$, so that no observable whatever can distinguish their coherent superposition from their incoherent mixture. This is just what is needed to make sense of an ignorance interpretation of *microphysical* probabilities; a definite (if unknown) event occurs on the microscopic scale precisely when the component states which enter into a coherent superposition evolve into disjoint states (because of some coupling with a measurement instrument). Indeed, any macroscopic event (whether or not a measurement) should be described in this way, so that quite generally it is from the transition to disjoint states that the probabilistic nature of classical experience is to derive.

In this connection, a cautionary note: in general an

incoherent mixture of states does not admit an ignorance interpretation of probability. This fact is intimately linked to the possibility of forming the coherent superposition of states which enter into the convex sum of the mixed state. In this way the component states which enter into a convex sum are not uniquely defined. What is required is that the component states form a *simplex* (Choquet and Meyer [1963]). The set of states, even pure states, on a C^*-algebra is much too large.

In general we must deal with reducible representations. An important class of representations which are "almost" irreducible are the **primary** representations (*i.e.* such that $\pi(\mathcal{A})''$ is a factor ; states which generate primary representations are called primary). It is a simple theorem (Hepp [1972]) that primary states which differ on global densities are disjoint. The importance of primary representations is this: they are associated with pure thermodynamic phases (Emch and Knopps [1970]). In particular, **extremal KMS** states are primary. These are the extremals of the convex set of KMS states at a given temperature; their interpretation as pure thermodynamic phases was first proposed by Ruelle ([1965]), who also showed (Ruelle [1967], [1969]) that these are the *only* equilibrium states which are dispersion free on global densities (see also Sewell [1986 Sec.4.4., App.B] for an extension of this result). As such they must yield distinct values to such observables and hence are also disjoint. Moreover, they form a *simplex* (Emch [1972 Th.2.2.15], and KMS states for different temperatures are also disjoint (Takesaki [1970]).

These results are about as good as one could hope for; the KMS condition is defined relative to a symmetry group on the quantum system (3.5.5) and there are good reasons to believe this condition is ultimately a consequence of locality and an ergodic property of the symmetry group (Kastler and Takesaki [1982]); in this sense the DPL theory of measurement is not so far removed from the present theory, except in the important respect that spatial averages, *not*

time averages, play the central rôle. Summarizing the DPL theory, Prosperi has declared:

> In conclusion the possibility seems to exist of getting out of the paradoxes connected with the occurrence of interference terms among macroscopically distinguishable states, assuming that physical observables uncompatible with the macroscopic quantities or at least with some privileged set of such quantities do not exist. Since however the idea that every self-adjoint operator (apart from superselection rules) corresponds to an observable, at least in principle, is quite naturally built in the mathematical structure of quantum mechanics, a consistent and logically satisfactory introduction of such a principle should require some kind of reformulation of the theory and perhaps some deep change in it (Prosperi [1971]).

It should be clear that the algebraic theory provides just such a reformulation of quantum mechanics and that, on the contrary, for systems of infinitely many degrees of freedom realizations of the algebra on some Hilbert space $\mathcal{H}$ appear quite naturally as *smaller* that $\mathcal{B}(\mathcal{H})$. One way of putting this situation is that macroscopic observables induce superselection rules on the representation of the algebra.

3.5.7. Theory of measurement; idealization, approximation, and realism.

There are two fundamental objections to the theory of measurement as outlined above. The first is that the thermodynamic representations cannot be interpreted in a realist way; that they explicitly describe infinite systems, of infinite particle number. As discussed in (3.5.2) this objection need not apply *in general* to non-Fock representations on the philosophy that the ultimate ontology of the world is a quantum field; nevertheless, this objection is ceded for these *particular* non-Fock representations. Of course, whether there exist alternative non-Fock representations which do not suffer from the unphysical characteristics of the thermodynamic representations (infinite volume, energy and mass) remains an open question.

The other problem, equally obvious, is in some sense even more intractable: KMS states, as all thermodynamic

equilibrium states, are time invariant. One can express this problem in an even more distressing manner as follows. Let $\alpha \in \mathrm{Aut}(\mathcal{A})$ be an automorphism of a C^*-algebra $\mathcal{A}$; for any state $\phi \in \mathcal{S}$ define the linear functional $\alpha^*[\phi]$: $\mathcal{A} \longrightarrow \mathbb{C}$ by $\langle \alpha^*[\phi]; A \rangle = \langle \phi; \alpha[A] \rangle$, for all $A \in \mathcal{A}$. It is easy to show that $\alpha^*(\phi)$ is a state and that α^* is a bijective *affine* map on $\mathcal{S}$. Therefore pure states are preserved under any automorphic time-evolution. As in the usual notion of superselection sectors, one can understand disjoint states as states which cannot be superposed to form pure states; therefore (there is a simple direct proof) disjoint states are also preserved by automorphic time-evolutions. This shows that if one insists that the time evolution is automorphic, there is no point in looking for non-Fock non-equilibrium representations, because the mechanism which is exploited to generate classical outcomes of experiments cannot be modelled as an automorphic evolution. We already gained some insight into this situation from the lattice theory of Section 2.2; we refer in particular to the concluding discussion of reduction theory ((2.2.5)).

One concludes that here, as in other applications, the thermodynamic limit is a tool for capturing certain features of the phenomenology, whilst at the same time introducing unphysical characteristics which are not present in finite-volume models. It is instructive to review some typical features of the finite and infinite volume analysis in connection with thermodynamic phenomena: in a phenomenological sense, physical systems have properties which are insensitive to the actual geometry of the system (intensive properties); phase transitions appear to be sharply defined with associated discontinuities in the thermodynamic potentials; distinct pure phases appear to coexist in equilibrium; and such systems appear to evolve irreversibly. None of these properties can be modelled within a finite-volume analysis: global averages of finite systems only converge to definite values as the number of degrees of freedom becomes infinite (Ruelle [1969]), thermodynamic variables are continuous (indeed analytic) for any finite system (Lebowitz [1968]), there exists a single,

unique KMS state (hence extremal) for a finite system at given volume, temperature, and mass, (hence no co-existence of distinct pure phases), and one can even show that metastable states (such as superheated or supercooled liquids) cannot be described in the standard finite theory (Fisher [1964]). To this list it seems one must add: a strict ignorance interpretation of probability can only be given in the infinite limit.

The analogy with Poincaré recurrences and the second law is helpful; in the classical theory a qualitative feature (reversibility and sensitivity to intial conditions) at the microscopic level cannot be eliminated in any finite model. Going to the thermodynamic limit one eliminates these features, but in the limit no *dynamic* behaviour can be described. This is more than a mere analogy - essentially the same mathematical technique is being used here as to destroy coherence in the quantum theory. In a sense the thermodynamic limit has long provided a theory of measurement for statistical mechanics (relating microscopic properties to thermodynamic ones); what is remarkable in the quantum case is not that the same representation theory yields a quantum thermodynamic description but that the space of states of this system should form a simplex.

The probabilistic behaviour of macroscopic systems therefore arises from the quantum indeterminism in the same way that irreversibility arises from a microscopic reversible system - as a theory of statics. How should we interpret this situation? The simplest idea is that the classical properties of the quantum system are to be understood as *idealizations* of certain properties of macroscopic systems; that is, any weakly coupled system of at least (say) 10^4 massive degrees of freedom over a time-scale of the order of the thermal relaxation time at usual temperatures (about 10^{-14} secs). Of course classical methods have been successfully used at much smaller scales than this, but our brief is the question of coherence, which is not a typical preoccupation of the chemist or crystallographer. These length scales (of order 10^{-8}m. for dilute systems) are

certainly sufficiently small to generate a classical field theory along the lines of Noll [1958], *i.e.* a continuum, fluid or thermodynamics.

However it is by no means clear that one can exclude the influence of systems past timelike or even spacelike to this volume in the measurement context; Einsteinian locality cannot be presumed in this field (so much is the moral of the Aspect experiments). Nevertheless we make the most simple assumption that the Cauchy problem is well-posed on a Cauchy volume for the system of indicated size, at least when taken in isolation (an assumption that is still true of each separated system in an EPR-type correlation). This leads to a considerable increase in the particle number of the "effective" system - taken with its local environment determined in this way, of order 10^{-6}m and 10^{10} degrees of freedom.

The drift of these remarks is that a "nice" decay of coherence with particle number would be "sufficiently" rapid such that systems of this order of scale have "approximately" classical properties. This raises the very serious problem of a residual coherence, however small and rapidly decaying, in the realm of macroscopic experience.

There is a long and honourable tradition in physics of simply identifying the infinite limit with an empirical context in which the limit is palpably not reached; in quantum physics the most prominant example is in scattering theory, where the formulation of theory in terms of asymptotic limits is not just a mathematical convenience but seems forced upon us by Haag's theorem. But here, as in so much else, the measurement problem imposes its own harsh discipline. If the question "when is a scattered particle free?" has limited philosophical interest (there may not, after all, exist any truly free particles), the notion that some "residual" coherence, however small, may contaminate the description of macroscopic events has profound significance for our notions of macroscopic realism.

In developing models of the measurement process which make use of infinite systems it is crucial that one obtains estimates on large but finite systems of which we suppose the infinite systems are idealizations. Failing this, at least let us be clear as to what length scale is supposed to be identified with the infinite volume limit. One might summarize the possible interpretations by means of the comparison of the several length scales which follow from general considerations - the cosmological, local environment, small-macroscopic, and microscopic - with the *two* qualitative distinctions available in the present theory - the local and the global. It is not even that the ambiguity concerns only the reference of the global properties of the system. There is a long tradition, particularly in the more mathematical literature, of interpreting the local observables (quasi-local algebra) with possible experiments that can be performed in the associated space-time region. This is surely of local environment or small-macroscopic length scale.

With so many problems besetting the use of infinite systems one might wonder whether after all we may not resign ourselves to dealing with the measurement problem with the more elementary techniques of finite systems. I feel the strongest counter-argument is this: solving the measurement problem on a realist basis must at the same time provide a description of the approach to equilibrium of many-particle systems, since that is what actually occurs in any physical experiment (and in all macroscopic phenomena). One might be optimistic in a fringe activity like measurement theory of finding a simple, finite resolution of the problem, but in statistical mechanics one has a major part of modern physics[13].

[13] It does not seem coincidental that in one popular example of a "simple, finite resolution", namely the Everett-Wheeler "many-worlds" interpretation, one is inexorably led to the question of how to define the ensemble of worlds which identify the world of our experience, as it appears to evolve, from the class of all worlds. This is statistical mechanics in disguise.

A more physical response is that one cannot formulate the dynamical approach to equilibrium of a closed and finite system in any *asymptotic* sense. That is because as time evolves, more and more particles enter the causal environment of the system studied and the effective particle number must increase as the cube of the time. Only in the infinite limit can this change in particle content leave the representation invariant and permit the use of asymptotic time limits.

Pursuing this line one might think that the proper approach is to consider finite *open* systems, in particular that the evolution of such a system which takes account of the influence of the causal environment should be a contraction semi-group (Davies [1976]). That it must be non-automorphic is clear. There is, further, a general theory due to Lewis and Thomas [1975], which shows us how to construct an (infinite) classical enviroment and an automorphic evolution acting on the finite system embedded in this enviroment which mimics the original evolution. Massen ([1982], [1984]), and Hannabus ([1984]) have provided quantized versions of the Lewis and Thomas dilation theory (in the bose and fermion case respectively); the latter, in particular, has shown that in the limit of an underdamped system the combined state of the system and environment become disjoint in the infinite time limit.

There are obvious attractions to this approach; there is a qualitative distinction available between the finite measuring apparatus and its infinite environment, and one can choose to consider the infinite environment as a convenient mathematical artifice for obtaining asymptotic limits on the "real" finite system. However it remains unclear how the disjointness obtained in the limit is to be understood in terms of this finite system. For all that, the synthesis of ideas contained in this theory of measurement is an impressive achievement.

The first algebraic theory of measurement tried rather to confront the original difficulty head on. This is the theory

of Klaus Hepp ([1972]), apparently an elaboration of ideas of M.Fierz and R.Jost. The idea is to find an infinite system $\mathcal{A}$ and a group of automorphisms α_i ($i\in\mathbb{Z}$), which evolve a state w "close to" some disjoint state w'. In this limit we know that no automorphism can actually map w onto w'; therefore for any i there will be some element $A\in\mathcal{A}$ such that $\langle\alpha_i^*[w]-w';A\rangle \neq 0$. Can we be sure that, whilst non-zero, such a cross-term can always be made small? The answer is negative: if this were so, *i.e.* for any ε we can find a number $N(\varepsilon)$ such that $\langle\alpha_n^*[w]-w';A\rangle < \varepsilon$ for $n > N(\varepsilon)$ and for any $A \in \mathcal{A}$, the sequence $\alpha_i^*[w]$ would be strongly convergent on the (disjoint) state w'. This means that, in any representation π in which $\alpha_n^*[w] \in \mathcal{G}_{j(\pi)}$ then so too would w' $\in \mathcal{G}_\pi$ (since this set is norm-closed by definition) so that w' could not be disjoint, a contradiction. Recalling, however, that $\mathcal{G}_\pi$ is the weak * closure of $\mathcal{G}_{j(\pi)}$ ((2.3.9)) one might hope to find a sequence $\{\alpha^*[w]\}$ which is weak * convergent on w', *i.e.* that for any ε and any *finite* set $\mathcal{M} \subset \mathcal{A}$ there is a number $N(\varepsilon,\mathcal{M})$ such that $\langle\alpha_n^*[w]-w';A\rangle < \varepsilon$ for $n > N(\varepsilon,\mathcal{M})$ and for all $A \in \mathcal{M}$.

It is not too surprising that if $w_{i,n} = \alpha_n^*[w_i]$, $i = 1,2$ are two such sequences, weakly convergent on two disjoint states w_i, then all cross-terms in any representation converge weakly to zero. More precisely, let π^n be a sequence of representations of $\mathcal{A}$ with $w_{i,n} = j_{\pi^n}(\phi_{i,n})$, $\phi_{i,n} \in \mathcal{H}_{\pi^n}$; then for any ε, $\mathcal{M}$, there is an $N(\varepsilon,\mathcal{M})$ with $(\phi_{1,n}, \pi_n(A)\phi_{2,n})_{\mathcal{H}_{\pi^n}} < \varepsilon$ for all $n > N(\varepsilon,\mathcal{M})$ and $A\in\mathcal{A}$ (Hepp [1972 Lemma 3]).

This result has a natural positivist interpretation (*cf.* (2.3.9)): that for any finite set of measurements (and the number of observations that have ever been or will ever be made is finite) the coherence terms vanish as $t \longrightarrow \infty$. Repudiating this positivist philosophy, Bell has remarked:

> While for any given observable one can find a time for which the unwanted interference is as small as you like, for any given time one can find an observable for which it is as big as you do *not* like (Bell [1975]).

All of the asymptotic models constructed by Hepp lead to asymptotic disjointness; the analogy, that

> The introduction of an asymptotic condition into measurement theory is as natural as elsewhere in microphysics, where S-matrix theory is sometimes considered the ultimate receptacle of all physics. (Hepp [1972]).

has already been found wanting. Incidentally, it should be clear that there is no general obstacle to the approach to disjointness in finite times if one is prepared to consider non-automorphic evolutions. For closed systems, however, this appears unacceptable for a variety of reasons. I shall not attempt to deal with this issue here.

There is another issue at stake, and that is the (usually implicit) shift in emphasis from the global to the local observables as the proper objects of macroscopic experience. Bell makes this shift explicitly, although he offers no justification for it. Neither does he offer any realist critique on the use of infinite systems. Yet this question is fundamental: it is, after all, a bonus that we should obtain a strict ignorance interpretation of microphysical probability (*i.e.* lack of coherence between states which describe different experimental outcomes with respect to all *microscopic* observables); this is even more than the projection postulate. Would any residual coherence here be observable? Of course we use data on macroscopic observables to deduce the expectation values of *some* microscopic observables; *these* should not be subject to any residual coherence. Bell's objection loses its purchase, if the observables for which the coherence "is as big as you do *not* like" are not macroscopic. But there remains the much more pressing question: when do the *macroscopic* properties of a measurement device change?

These ambiguities weaken a widely held interpretation of the various models that have been proposed (Frigerio [1976]; Whitten-Wolfe and Emch [1976]; Hannabus [1984]; Zurek [1981]), that the coherence is never really lost, but propagates out over vast regions of space where it is no longer observable. The difficulty is that the macroscopic properties are globally defined, typically as the algebra of observables at infinity. One wants simultaneously to assert

that the global observables describe small macroscopic systems, yet there exist local observables which describe properties pertaining to "vast" regions of space. One does not have enough qualitatively distinguished length scales in the theory.

One might conclude that the macroscopic observables of a system play a merely technical rôle in the present approach to measurement theory. Yet this is to abandon the fundamental intuition that directly observable phenomenology - both with respect to probabilistic behaviour and to thermodynamic behaviour - arises as collective phenomena which can be directly described by macroscopic observables. There is surely something inconsistent with the view that observed thermodynamic properties be associated with classical observables in the context of statistical mechanics yet be relegated to a purely technical rôle in measurement theory.

Considering this difficulty, one feels a certain exasperation. The association of large length scales, on the one hand with macroscopic properties, on the other hand with an unobservable residual coherence, is too simplistic on both scores. It is rather a matter of the kind of observable: whether approximating a quasi-equilibrium collective property, or (possibly long-range) correlations among microscopic properties. And it is in the latter that we might hope to dispose of any residual coherence. This is only to say that the latter, regardless of length scale, should not count as macroscopic properties. The difficulty may be apparent rather than real.

This conclusion is reinforced by a closer analysis of how, in fact, such residual coherence might be invasive at the macroscopic level. The point is, that although we have no idea what such macroscopic quantum coherence (MQC) would look like, it is the sort of thing which falls off extremely rapidly with time and with particle number of the effective system. There is an extremely strong disturbance effect of measurement: the coupling of one experimental system to

another enormously increases the complexity of the total effective system. In this way the von Neumann infinite regress that appears to follow from elementary quantum mechanics is rapidly damped.

One of the most important phenomenological considerations is the nature of subjective experience itself. The subjective experience of radiation is a very sensitive measurement device (observers have been known to detect light levels involving only a few quanta). The effective system that we must consider has, however, a very large number of degrees of freedom (of the order of the human brain, say 10^{28}). If the fall-off in coherence is fast enough for a system of this scale the subjective experience of MQC will be of negligible duration in comparison to the time-scale of subjective experience. This will apply *a fortiori* to the *perceptible* behaviour of laboratory instruments.

Nevertheless, there are indirect methods; in particular it may be possible to exploit the Bell inequality as a test for MQC in the behaviour of certain types of macroscopic systems. Consider a macroscopic system with some property Q which, on measurement, is always found to have one of two values ± 1. Fix a time interval τ and time t and define the observables Q^t_{ij} , $i,j = 0,1,2$, where $Q^t_{ij} = Q(t+i\tau)Q(t+j\tau)$. Prepare an ensemble of such systems in a similar way at time t_0 and define the ensemble average $\langle Q^t_{ij}\rangle$; if now we assume that the measurement of $Q(t+i\tau)$ does not effect the value of $Q(t+j\tau)$ (this is equivalent to what is usually, and misleadingly, called the locality assumption in standard treatments of the Bell inequality), then the inequality

$$|\langle Q^t_{13}\rangle+\langle Q^t_{23}\rangle+\langle Q^t_{14}\rangle-\langle Q^t_{24}\rangle| \leq 2$$

will hold if $Q(t+i\tau)$ always has a definite value (whether or not it is measured; see, *e.g.* Clauser and Shimony [1978]). The Bell inequality therefore provides, under the stated conditions, a phenomenological test for discovering MQC. The theoretical problem of selecting a "suitable" class of systems, and in particular of justifying the condition on the independence of the $Q(t+j\tau)$ and on the choice of $t-t_0$, τ, is, however, somewhat daunting. We refer to Legett [1985]

for a proposal in which the magnetic flux trapped in one of two potential minima of a SQUID (superconducting quantum interference device) and oscillating between them via quantum tunelling olays the rôle of the observable Q above.

On estimates: Frigerio [1976] considered a finite apparatus coupled to an infinite environment as a finite volume of a 1-dimensional spin 1/2 lattice. In this model local observables were again considered as measurable; the estimate t^{1-n} was obtained (with n the number of lattice sites) for the ratio of the interference terms to the difference of the expectation values of such observables in a suitable class of states. Obviously the most pressing need is to produce estimates for more realistic models for their associated *macroscopic* observables. The indications are that for these observables we may expect the fall off to be so rapid as to be effectively instantaneous. If successful in this task, one will have the curious situation that these effects will occur *whether or not* they are considered relevant to the measuring process, as remarked by Hannabus [1984].

We do not yet have a satisfactory measurement theory. But the fundamental ideas seem to be all in place. The problems appear to be well-defined mathematical ones, susceptible to analysis in the usual way. The fundamental ideas, moreover, appear to be physically compelling, and not merely the exploitation of mathematical finesse. In support of this claim I will conclude with a curiosity, a fragment of a personal memo, written at about the same time that Segal was perfecting the representation theory of C^*-algebras:

> When you start out to measure the property of one (or more) atom, say, you get, for example, a spot on a photographic plate which you then interpret. But such a spot is really only more atoms and so in looking at the spot you are again measuring the properties of atoms, only now it is more atoms. What can we expect to end up with if we say we can't see many things about one atom precisely, what in fact can we see? Proposal: *Only those properties of a single atom can be measured which can be correlated (with finite probability) (by various experimental arrangements) with an unlimited number of atoms.*
> R.Feynman, private memo, 1946 (emphasis mine, quoted in Schweber [1986p.463]).

Conclusions

A realist reconstruction of quantum theory is possible. It is necessary to introduce the primitive concepts of *measure*, *observable*, and *state*, in terms of which physical entities are described. We identify the observables with entities in the world, so that a description of the world is given by defining the state of the observables. This description must be such that when a set of observables is described by numerical magnitudes, then the usual algebraic relationships obtain. This leads to the classical theory under mild conventionalist assumptions, when we insist that maximal desciptions of all observables always exist. Relinqushing this requirement one has an abstract algebra which with similar assumptions is a C^*-algebra. A *quantum field theory* is such an abstract system, together with a way of associating sub-systems with a field of subsets of spacetime, such that it is possible to formulate the Cauchy problem. A spacetime symmetry group automatically defines a continuous family of * automorphisms of the algebra.

The quantization of linear classical systems is not directly formulated in these terms (nor is it clear that such a procedure should have a realist interpretation), but through the notion of a Weyl algebra over a linear classical phase space. The central feature of QFT, namely that there exist inequivalent representations of the abstract C^*-algebra, then shows that *thermodynamic properties* are characterized by a representation. The primitive notion of measure is shown to admit a probabilistic interpretation for such properties. The measurement problem is resolved in the sense that coherent superpositions of states in which thermodynamic properties differ do not exist. The remaining problem is similar to that in the realist interpretation of the thermodynamic limit: that this limit is an *idealization.*

Linear relativistic quantum fields in the usual formalism are described as a representation of the generators of the Weyl and Clifford algebras over a classical phase space. This is exactly the same procedure as in NRQFT, and in all cases it is necessary, in order to define an action of the field on the Fock space over the classical phase space, to select a real symplectic (Weyl) or orthogonal (Clifford) transformation on the classical phase space (a *positive compatible complex structure*), which enables us to look at the classical evolution as at once a group of symplectic (orthogonal) transformations, and also a group of *unitary* transformations on this same space.

The non-relativistic and relativistic quasi-free theories differ in this choice of complex structure. That is the *only* difference at this level in the two theories. In particular both fermion and bosonic theories have exactly the same complex structure, and in an identical way describe the negative frequency states as positive energy anti-particle states. One can turn this procedure around, and starting off with this complex structure on the classical solution space, now regarded as the 1-particle Hilbert space, perform a canonical second quantization: there is no need for normal ordering, and the equivalence of field and many-particle system is exactly as in the NRQT. The charge conjugation operator becomes unitary in both 1-particle theory and field theory.

Imposing a Born interpretation does two things: first, the natural parameterization of the classical solution space (or 1-particle Hilbert space) is then non-locally related to the covariant parameterization used previously, and second it is not covariant. But in this form (position space) the canonical creation and annihilation fields are q-causal, and an exact q-local correspondence exists between field theory and 1-particle theory for all q-local observables.

From this point of view, it is merely a curiosity that one

may recover the same theory by canonically second quantizing the identical system, but using the *natural* complex structure, and then imposing the previous particle interpretation (by normal ordering with respect to the action of the fields on the Fock space over the solution manifold equipped with the *particle* complex structure). The creation and annihilation operators defined with respect to the natural complex structure are automatically linear combinations of creation and anihilation operators defined with respect to the particle complex structure. In this way we obtain an interpretation of the prescriptions of the Dirac hole theory, and we see that the Dirac vacuum is what the true particle vacuum looks like from the point of view of the natural complex structure. However the present interpretation applies to the bosonic case as well.

We now note two things: first, the natural complex structure is c-local and is invariant under *any* linear interaction (as a symplectic or orthogonal transformation on the classical solution manifold); second, the particle complex structure is non c-local and is *not* invariant in this way.

The interacting theory could be formulated as a q-local interaction in the canonical way of NRQM; the resulting theory would *not* be covariant, and would *not* lead to creation and annihilation processes. Instead the conventional theory c-locally couples to the natural creation and annihilation operators *and then* expresses the action of the resulting observables in terms of the particle complex structure. Because the latter does not in general commute with the Hamiltonian, such operators lead to pair creation and annihilation processes. These are *pair* processes, because the interactions are assumed invariant under rotations in the *natural* complex structure. Because in this approach the (non)local action of the complex structure is tied directly to (non)causality, we learn that the creation and annihilation operators defined by the natural and particle complex structures are c-causal and q-causal respectively.

BIBLIOGRAPHY

Albert A.A. [1934]: "On a certain algebra of quantum mechanics", *Ann. of Math.* **35**, 65-73.

Anderson C. [1932]: "The apparent existence of easily deflectable particles", *Science* **76**, 238

Araki H. [1969] "Local quantum theory 1", *in* Jost [1969].

Araki H.,and E. Woods [1963]: "Representations of the canonical commutation relations describing a non -relativistic infinite free Bose gas", *J. Math. Phys.* **4**, 637-62.

Araki H., K. Hepp and D. Ruelle [1962]: "On the asymptotic behaviour of Wightman functions in space-like directions", *Hel. Phys. Acta* **35**, 164-74.

Ashtekar A. and A. Magnon [1975]: "Quantum fields in curved space-times", *Proc. Roy. Soc. London A* **346**, 375-94.

Bargmann V. [1947]: "Irreducible unitary representations of the Lorentz group", *Ann. of Math.* **48**, 568-640.

Bargmann V. [1954]: "On unitary ray representations of continuous groups", *Ann. of Math.* **59**, 1-46.

Bargmann V. [1961]: "On a Hilbert space of analytic functions and an associated integral transform Part 1", *Comm. Pure Appl. Math.* **14**, 187-214.

Bargmann V. [1964]: "Note on Wigner's theorem on symmetry operations", *J. Math. Phys.* **5**, 862-68.

Bargmann V., and E. Wigner [1948]: "Group theoretical discussion of relativistic wave equations", *Proc. Nat. Acad. Sci.* **34**, 211-23.

Barut A., and R Raczka [1977]: *Theory of Group Representations and Applications*, PWN, Polish Scientific Publishers, Warsaw.

Bekenstein J. [1973]: "Black holes and entropy", *Phys. Rev. D* **7**, 2333-46.

Bekenstein J. [1983]: "Entropy bounds and the second law for black holes", *Phys. Rev. D* **27**, 2262-70.

Bell J. [1975]: "On wave packet reduction in the Colemann - Hepp model", *Helv. Phys. Acta* **48**, 93-8.

Beltrametti E. and G. Cassinelli [1981]: *The Logic of Quantum Mechanics*, Addison-Wesley, Reading, Mass.

Bhaba H. [1945]: "Relativistic wave equations for the elementary particles", *Rev. Mod. Phys.* **17**, 203-9.

Bhaba H. and Harish-Chandra [1946]: "On the fields and equations of motion of point particles", *Proc. Roy. Soc. A* **185**, 250-68.

Birkhoff G., and J. von Neumann [1936]: "The logic of quantum mechanics", *Ann. of Math.* **37**, 823-843.

Birrel N., and P. Davies [1982]: *Quantum Fields in Curved Space*, CUP, Cambridge.

Bjorken J. and S. Drell [1964]: *Relativistic Quantum Mechanics*, McGraw-Hill, New York.

Bjorken J. and S. Drell [1965]: *Relativistic Quantum Fields*, McGraw-Hill, New York.

Bloch Z. [1934]: "Die physikalische Bedeutung mehrerer Zeiten in der Quantenelektrodynamik", *Phys. Zeit. Sowjetunion* **5**, 301-15.

Bohr N. [1918] "On the quantum theory of line-spectra", *Mem. de l'Acad. Roy. des Sci. et des Lett. de Danemark,* **4.1**, 1-3, *also in* van der Waerden [1967 p.95-137].

Bohr N., H. Kramers and J. Slater [1924]: "The quantum theory of radiation", *Phil. Mag.* **47**, 785-802, *also in* van der Waerden [1967 p.159-176].

Bohr N., and L. Rosenfeld [1933]: "Zur Frage der Messbarkeit der Electromagnetischen Feldgrossen", *Dann. Mat. Fys. Medd. X11*, **8**, *English translation in* Cohen and Stachel [1979].

Bohr N., and L. Rosenfeld [1950]: "Field and charge measurements in quantum electrodynamics", *Phys. Rev.* **78**, 794-98, *also in* Cohen and Stachel [1979 p.401-12].

Bongaarts P. [1970]: "The electron-positron field, coupled to external electromagnetic potentials, as an elementary C^*- algebra theory", *Ann. of Phys.* **56**, 108-139.

Bongaarts P. [1972]: "Linear fields according to I.E. Segal", *in* Streater [1972 p.187-208].

Borchers H., [1967] "On the theory of local observables" *in* Lurcat [1967 p.3-39]

Born M. [1924]: "Über Quantenmechanik", *Zeit. f. Phys.* **26**, 379-95, *English translation in* van der Waerden [1967 p. 181-98].

Born M., and P. Jordan [1925]: "Zur Quantenmechanik", *Zeit. f. Phys.* **34**, 858-88.

Born M., W. Heisenberg and P. Jordan [1926]: "Zur Quantenmechanik II", *Zeit. f. Phys.* **35**, 557-615, *English translation in* van der Waerden [1967, 321-385].

Born M., and K. Fuchs [1938]: "On fluctuations in electromagnetic raditation", *Proc. Roy. Soc. A* **170**, 252, **172** 465.

Born M., and K. Fuchs [1940]: "The mass-centre in relativity", *Nature* **145**, 587.

Bose N. [1924]: "Plancksche Gesetz und Lichtquantenhypothese", *Zeit. f. Phys.* **26**, 178-81.

Bothe W, and H. Geiger [1924]: "Ein Weg zur experimentellen Nachprufung der Theorie von Bohr, Kramers und Slater", *Zeit. f. Phys.* **26**, 44.

Breit G. [1928]: "An interpretation of Dirac's theory of the electron", *Proc. Nat. Acad. Sci.* **14**, 553-59.

Bromberg J. [1976]: "The concept of particle creation before and after quantum mechanics", *Historical Studies in the Physical Sciences*, **7**, 161

Bromberg J. [1977]: "Dirac's quantum electrodynamiks and the wave-particle equivalence", *in* Wiener [1977 p. 147-57].

Brown H., and R. Harre [1988]: *The Philosophy of Quantum Field Theory*, Clarendon Press, Oxford.

Carruthers P., and M Nieto [1968]: "Phase and angle variables in quantum mechanics" *Rev. Mod. Phys.* **10**, 411-40.

Case K. [1954]: "Some generalizations of the Foldy-Wouthuysen transformation", *Phys. Rev.* **95**, 1323-1328.

Casimir H. [1948]: *Proc. K. Ned. Akad. Wet.* **51**, 793.

Choquet G. [1969]: *Lectures on Analysis, Vol I, II, and III*, Benjamin, New York.

Choquet G. and P. Meyer [1963]: "Existence et unicité des représentations integrales dans les convexes compacts quelconques", *Ann. Inst. Fourier* (Grenoble) **13**, 139-54.

Christenson S. [1984]: *Quantum Theory of Gravity, essays in honour of the 60th birthday of Bryce S DeWitt*, Adam Hilger, Bristol.

Clauser J. and A. Shimony [1978]: "Bell's theorem: experimental tests and implications", *Rep. Prog. Phys.* **41**, 1881.

Cohen R., and J. Stachel [1979]: *Selected Papers of Leon Rosenfeld, Boston Studies in the Philosophy of Science* **21**.

Compton A. and A. Simon [1925]: "Directed quanta of scattered x-rays", *Phys. Rev.* **26**, 289-299.

Cook J. [1953]: "The mathematics of second quantization", *Trans. Am. Math. Soc.* **74**, 222-245.

Currie D., T. Jordan, and E. Sudershan [1963]: "Relativistic invariance and Hamiltonian theories of interacting particles", *Rev. Mod. Phys.* **35**, 350-75.

Cushing J. [1988]: "Foundational problems in quantum field theory", *in* Brown and Harre [1988 p.25-39].

Daneri A., A. Loinger, and G. Proseperi [1962]: "Quantum theory of measurement and ergodicity conditions", *Nucl. Phys.* **33**, 297-319.

Darwin C. [1928]: "Wave equation of the electron", *Proc. Roy. Soc. A* **118**, 654-680.

Davies B. [1976]: *Quantum Theory of Open Systems*, London, Academic Press.

Davies B., and J. Lewis [1970]: "An operational approach to quantum probability", *Comm. Math. Phys.* **17**, 239-260.

Davies P. [1984]: "Particles do not exist" *in* Christensen [1984 p.66-77].

Davies P., and S. Fulling [1977a]: "Quantum vacuum energy in two dimensional space-times", *Proc. Roy. Soc. A*, **354**, 59-77.

Davies P., and S. Fulling [1977b]: "Radiation form moving mirrors and from black holes", *Proc. Roy. Soc. A*, **356**, 237-257.

de Broglie L. [1925]: "Recherches sur la theorie des quanta", *Ann. de Phys.* **3**, 22-128.

Debye P. [1910]: "Der Wahrscheinlichkeitsbegriff in der Theorie der Strahlung", *Ann. der Phys.* **33**, 1427-34.

Deliyannis P. [1975]: "Imbedding of Segal systems", *J. Math. Phys.* **16**, 163-170.

d'Espagnat B. [1971] (ed.) *Foundations of Quantum Mechanics; proceedings of the International School of "Enrico Fermi", Number 49*, Academic Press, New York.

Deutsch D., and P.Candelas [1979]: "Boundary effects in quantum field theory", *Phys. Rev. D* **20**, 3063-80.

Deutsch D., and Najmi [1983]: "Construction of states for quantum fields in nonstatic space-times", *Phys. Rev. D* **28**, 1907-15.

de Witt B. [1979]: "Quantum gravity: the new synthesis", *in* Hawking and Israel [1979 p.680-745].

Dirac P. [1925]: "The fundamental equations of quantum mechanics", *Proc. Roy. Soc. A* **109**, 642-53.

Dirac P. [1926a]: "Quantum mechanics and a preliminary investigation of the hydrogen atom", *Proc. Roy. Soc. A* **110**, 561-69.

Dirac P. [1926b]: "Relativity quantum mechanics with an application to Compton scattering" *Proc. Roy. Soc. A* **111**, 405-21.

Dirac P. [1926c]: "The quantum algebra", *Proc. Cam. Phil. Soc.* **23**, 412-8.

Dirac P. [1926d]: "On the theory of quantum mechanics" *Proc. Roy. Soc. A* **112**, 661-77.

Dirac P. [1926e]: "The Compton effect in wave mechanics", *Cam. Phil. Soc.* **23**, 500-7.

Dirac P. [1926f]: "The physical interpretation of the quantum dynamics", *Proc. Roy. Soc. A* **113**, 621-641.

Dirac P. [1926g]: "On the elimination of nodes in quantum mechanics", *Proc. Roy. Soc. A* **111**, 281-305.

Dirac P. [1927a]: "The quantum theory of the emission and absorption of radiation", *Proc. Roy. Soc. A* **114**, 243-65.

Dirac P. [1927b]: "The quantum theory of dispersion", *Proc. Roy. Soc. A* **114**, 710-28.

Dirac P. [1928a]: "The quantum theory of the electron", *Proc. Roy. Soc. A* **117**, 610-24.

Dirac P. [1928b]: "The quantum theory of the electron", *Proc. Roy. Soc. A* **118**, 351-61.

Dirac P. [1929]: "A theory of electrons and protons", *Proc. Roy. Soc. A* **126**, 360-5.

Dirac P. [1931]: "Quantized singularities in the electromagnetic field", *Proc. Roy. Soc. A* **133**, 60-72.

Dirac P. [1932]: "Relativistic quantum mechanics", *Proc. Roy. Soc. A* **136**, 453-465.

Dirac P. [1934]: "Discussion of the infinite distribution of electrons in the theory of the positron", *Proc. Cam. Phil. Soc.* **30**, 150-63.

Dirac P. [1935]: "Lectures on quantum mechanics", Princeton Lecture notes (unpublished).

Dirac P. [1936]: "Relativistic wave equations", *Proc. Roy. Soc. London A* **155**, 447-459.

Dirac P., V. Fock and Podolsky [1933]: "On quantum electrodynamics", *Phys. Zeit. d. Sowjetunion* **2**, 468-79.

Dixmier P. [1957]: *Les C^*-Algèbres et Leurs Représentations*, Paris, Gathier-Villars.

Doplicher S., D. Kastler, and E. Stormer [1969]: "Invariant states and asymptotic abelianess", *J. Func. Anal.* **3**, 419-433.

Dorling J. [1986]: "Schrödinger's original interpretation of the Schrödinger equation: a rescue attempt" *in* Kilminster [1987 p.16-40].

Duffin R. [1938]: "On the characteristic matrices of covariant systems", *Phys. Rev.* **54**, 1114.

Eddington A. [1926]: *The Internal Constitution of the Stars*, CUP, Cambridge.

Edwards C. [1970]: "The operational approach to algebraic quantum theory 1", *Comm. Math. Phys.* **16**, 207-230.

Ehrenfest P. [1925]: "Energieschwankungen ins Strahlungsfeld oder Kristallgitter bei Superposition quantisierter Eigenschwingungen", *Zeit. f. Phys.* **34**, 362-73.

Ehrenfest P. [1932]: "Einige die quantummechanik betreffende Erkundigungsfragen", *Zeit. f. Phys.* **78**, 555-9.

Ehrenfest P. and U. Trkal [1921]: Ableitung des Dissoziationsgliechgewichtes aus der Quantentheorie und darauf beruhende Berechnung der chemischen Konstanten", *Ann. der. Phys.* **65**, 609

Eidelheit M. [1940]: "On isomorphisms of rings of linear operators", *Studia Mathematica* **9**, 97-105.

Einstein A. [1909]: "Zum gegenwartigen Stand des Strahlungsproblems", *Phys. Zeit.* **10**, 185-93.

Einstein A. [1917]: "Quantentheorie der Strahlung", *Phys. Zeit.* **18**, 121-8, *English translation in* van der Waerden [1967 p.63-77].

Einstein A. [1924]: "Quantentheorie des einatomigen idealen Gases", *Berliner Berichte*, 261-267.

Einstein A. [1925]: "Quantentheorie des einatomigen idealen Gases", *Berliner Berichte*, 3-14.

Emch G., [1972] *Algebraic Methods in Statistical Mechanics and Quantum Field Theory*, Wiley Interscience, New York.

Emch G., and M. Guenin [1966]: "Gauge invariant formulation of the BCS-model", *J. Math. Phys.* **7**, 915-921.

Emch G., and H. Knopps [1970]: "Pure thermodynamic phases as extremal KMS states", *J. Math. Phys.* **11**, 3008-18.

Fell J. [1960]: "The dual spaces of C^*-algebras", *Tr. Am. Math. Soc.* **94**, 365-403.

Feshbach H., and F. Villars [1958]: "Elementary relativistic wave mechanics of spin 0 and spin 1/2 particles", *Rev. Mod. Phys.* **30**, 24-45.

Fierz M. [1939]: "Über die relativistische Theorie kraftefreier Teilchen", *Helv. Phys. Acta* **12**, 3-37

Finkelstein R. [1948]: "The quantization of unitary field theories", *Phys. Rev A.* **74**, 1563.

Finkelstein D., J. Jauch, and D. Speiser [1963]: "Quaternionic representations of compact groups", *J. Math. Phys.* **4**, 136-140.

Finkelstein D., J. Jauch, S. Schiminovitch, and D. Speiser [1962]: "Foundations of quaternion quantum mechanics", *J. Math. Phys.* **3**, 207-220.

Finkelstein D., J. Jauch, S. Schiminovitch, and D. Speiser [1963]: "Principle of general Q-covariance", *J. Math. Phys.* **4**, 788-96.

Fisher M. [1964]: "The free energy of macroscopic systems", *Arch. Rat. Mech. Annal.* **17**, 377-410.

Fleming G. [1965]: "Covariant position operators, spin, and locality", *Phys. Rev B* **137**, 188-97.

Fleming G. [1966]: "A manifestly covariant description of dynamical variables in relativistic quantum mechanics", *J. Math. Phys.* **7**, 1959-81.

Fleming G. [1985]: "Towards a Lorentz-invariant quantum theory of measurement", Pennsylvania State University preprint, to appear in *Mini-Course and Workshop in Fundamental Physics*, University of Puerto Rico Press, Puerto Rico.

Fock V. [1928]: "Verallgemeinerung und Lösung der Diracschen statistischen Gleichung", *Zeit. f. Physik* **49**, 339-357.

Fock V. [1929]: "Über den Begriff der Geschwindigkeit in der Diracschen Theorie des Elektrons", *Zeit. f. Phys.* **55**, 127-40.

Fock V. [1932]: "Konfigurationsraum und zweite Quantelung", *Zeit. f. Phys.* **21**, 78-89.

Fock V. [1933]: "Zur theorie der positronen", *Comptes Reundus (Doklady) de l'Academie des Science de l'URSS* **3**, 267-71.

Focker A. [1929]: "On the space-time geometry of a moving rigid body", *Rev. Mod. Phys.* **21**, 406-408.

Foldy L. [1956]: "Synthesis of covariant particle equations", *Phys. Rev.* **102**, 568-581.

Foldy L. and S. Wouthuysen [1950] "On the Dirac theory of spin 1/2 particles and its non-relativistic limit", *Phys. Rev.* **78**, 29-36.

Fonda L., and G. Ghirrardi [1968]: "Some remarks about the definition of position operator for relativistic elementary systems", *Nouvo Cimento* **56**, 1094-8.

Fonda L., and G. Ghirrardi [1970]: *Symmetry Principles in Quantum Physics*, Marcel Dekker, New York.

Foulis D. [1960]:"Baer * - semigroups", *Proc. Amer. Math. Soc.* **11**, 648-654

Fredenhagen K. and J. Hertel [1981]: "Local algebras of observables and pointlike localized fields", *Comm. Math. Phys.* **80**, 555-61.

French S. [1984]: *Identity and Individuality in Classical and Quantum Physics*, Ph.D thesis, University of London (unpublished).

Friedrichs K. [1953]: *Mathematical Aspects on the Quantum Theory of Fields*, Wiley-Interscience, New York.

Frigerio A. [1976]: "Quasi-local observables and the problem of measurement in quantum mechanics", *Ann. Inst. H. Poincare* **A21**, 259-70.

Fulling S. [1973]: "Nonuniqueness of canonical field quantization in Riemannian space-time", *Phys. Rev. D*, **7**, 2850-62.

Fulling S. [1983]: *Gauge Theory and Gravitation: Lecture Notes in Physics No. 176* p.101-6, Springer, Berlin.

Fulling S. [1984]: "What have we learned from quantum field theory in curved space-time?", *in* Christensen [1984 p.42-52].

Furry and Openheimer [1932]: "On the theory of the electron and positive", *Phys. Rev.* **45**, 245-62, 343-4.

Gel'fand I., and M. Naimark [1943]: "On the imbedding of normed rings into the ring of operators in Hilbert space", *Rec. Math. (Mat. Sbornik) N.S.* **12**, 197-213.

Gel'fand I., and M. Naimark [1947]: "Unitary representations of the Lorentz group", *Bull. Acad. Sci. URSS. Ser. math.* **11**, 411-504.

Gel'fand I., and D. Rykov [1943]: "Irreducible unitary representations of locally compact groups", *Rec. Math. (Mat. Sbornik) N.S.* **13**, 301-316.

Gel'fand I., and G. Shilov [1958]: *Generalized Functions*, English translation 1964, Academic Press, New York.

Gibbons G. [1979]: "Quantum field theory in curved spacetime", *in* Hawking and Israel [1979 p.639-79].

Glauber R. [1963]: "Coherent and incoherent states of the radiation field", *Phys. Rev.* **131**, 2766-88.

Gleason A. [1957]: "Measures of the closed subspaces of a Hilbert space", *J. Math. Mech.* **6**, 885-893.

Glimm J. [1961]: "Type 1 C^*-algebras", *Ann. of Math.* **73**, 572-612.

Glimm J., and A. Jaffe [1972]: "Boson quantum field models", *in* Streater [1972].

Glimm J., and A. Jaffe [1981]: *Quantum Physics*, Springer -Verlag, New York.

Godement R. [1951]: "Sur la theorie des representations unitaires", *Ann. of Math.* **53**, 68-124.

Goldstein H. [1950]: *Classical Mechanics*, Addison-Wesley, Reading, Mass.

Gonzales J., and H. Wergeland [1973]: "Einstein-Lorentz's formula for the fluctuations of electromagnetic energy", *Det Kongelige Norske Videnskabers Skelskab No. 4*.

Gordan W. [1926]: "Der Comptoneffekt nach der Schrödingerschen Theorie", *Zeit. f. Phys.* **40**, 117-33.

Greenberg O. [1961]: "Generalized free fields and models of local field theory", *Ann. Phys.* **16**, 158-175.

Gudder S. [1973]: "Quantum logics, physical space, position observables and symmetry", *Rep. Math. Phys.* **29**, 249-64.

Gudder S. [1977]: "Four approaches to axiomatic quantum mechanics", *in* Price and Chissick [1977 p.247-76].

Guenin M. [1966]: "On the interaction picture", *Commun. Math. Phys.* **3**, 120-32.

Guth E. [1927]: "Zur Ableitung der Schrödingerschen Wellengleichung", *Zeit. f. Phys.* **41**, 235-38.

Haag R. [1955]: "On quantum field theories", *Det Kong. Dans. Vid. Sels.* **29**, 1-37.

Haag R. [1962]: "The mathematical structure of the Bardeen-Cooper-Schrieffer model", *Nuovo Cimento* **25**, 287-298.

Haag R., N. Hugenholtz, and M. Winnink [1967]: "On the equilibrium states in quantum statistical mechanics", *Commun. Math. Phys.* **5**, 215-36.

Haag R., and E. Trych-Pohlmeyer [1974]: "Stability and equilibrium states", *Commun. Math. Phys.* **38**, 173

Haag R. and D. Kastler [1964]: "An algebraic approach to quantum field theory", *J. Math. Phys.* **5**, 848-861.

Haag R., and B. Schroer [1962]: "Postulates of quantum field theory", *J. Math. Phys.* **3**, 248-256.

Haag R. and J. Swieca [1965]: "When does a quantum field theory describe particles?", *Commun. Math. Phys.* **1**, 308-20.

Halpern F. [1968]: *Special Relativity and Quantum Mechanics*, Prentice-Hall, N.J.

Hanlé P. [1977]: "Erwin Schrödinger's reaction to Louis de Broglie's thesis on the quantum theory", *Isis*, **68**, 606.

Hannabus K. [1984]: "Dilations of a quantum measurement", *Helv. Phys. Acta* **57**, 610-20.

Hareesh-Chandra [1946]: "On the equations of motion of point particles", *Proc. Roy. Soc. A* **185**, 269-87.

Harish-Chandra [1947]: "Infinite irreducible representations of the Lorentz group", *Proc. Roy. Soc. A* **189**, 372-401.

Harish-Chandra [1953]: "Representations of a semi-simple Lie group on a Banach space I", *Trans. Amer. Math. Soc.* **75**, 185-243.

Hawking S. [1974]: "Black hole explosions?", *Nature*, **248**, 30.

Hawking S. [1975]: "Particle creation by black holes", *Commun. Math. Phys.* **43**, 199-220.

Hawking S. [1977]: "Zeta function regularization of path integrals in curved spacetime", *Comm. Math. Phys.* **55**, 133-48.

Hawking S., and G. Ellis [1973]: *The Large Scale Structure of Spacetime*, CUP, Cambridge.

Hawking S., and W. Israel [1979] (eds): *General Relativity, an Einstein Centenary Survey*, CUP, Cambridge.

Heisenberg W. [1925]: "Über quantentheoretische Umdeutung kinematischer und mechanischer Beziehungen", *Zeit. f. Phys.* **33**, 879-93, *English translation in* van der Waerden [1967 p.261-76].

Heisenberg W. [1927]: Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik", *Zeit. f. Physik*, **43**, 172-198.

Heisenberg W. [1928]: letter to Pauli, July 31, *in* Pauli [1979 vol 1 p.466].

Heisenberg W. [1931]: "Über Energieschwankungen in einem Strahlungsfeld", *Berichte Sachsischen Akad. d. Wissen. z. Leipzig, Math-Phys. Kl.* **83**, 1-9.

Heisenberg W. [1934]: "Bemerkungen zur Diracschen Theorie des Positrons", *Zeit. f. Phys.* **90**, 209-231, **92**, 692.

Heisenberg W. and W. Pauli [1929]: "Zur Quantendynamik der Wellenfelder", *Zeit. f. Phys.* **56**, 1-61.

Heisenberg W. and W. Pauli [1930]: "Zur Quantendynamik der Wellenfelder 2", *Zeit. f. Phys.* **59**, 168-90.

Heitler W. [1943]: "On the particle equations of the meson", *Proc. Roy. Irish Acad.* **49**, 1-28.

Heitler W. [1957]: *The Quantum Theory of Radiation*, 3rd. Edition, OUP, Oxford.

Henley E., and W. Thirring [1962]: *Elementary Quantum Field Theory*, McGraw-Hill, New York.

Hepp K. [1972]: "Quantum theory of measurement and macroscopic observables", *Hel. Phys. Acta* **45**, 237-248.

Hermann R. [1966]: *Lie Groups for Physicists*, W. A. Benjamin, New York.

Heywood P. and M. Redhead [1985]: "Non-locality and the Kochen - Specker paradox", *Found. Phys.* **13**, 481-499.

Hinton K. [1983]: "Particle detectors in Rindler and Schwarzchild space-times", *J. Math. Phys. A: Math. Gen.* **16**, 1937-46.

Holdsworth D., and C. Hooker [1983]: "A crticial survey of quantum logic", in *Logic in the Twentieth Century, Scientia* 127-246.

Hooker C. [1975]: *The Logico-Algebraic Approach to Quantum Mechanics Vol. 1; University of Western Ontario series in Philosophy of Science*, D. Reidel, Boston.

Hooker C. [1979a]: *The Logico-Algebraic Approach to Quantum Mechanics Vol. 2; University of Western Ontario series in Philosophy of Science*, D. Reidel, Boston.

Hooker C. [1979b]: *Physical Theory as Logico-Operational Structure; University of Western Ontario series in Philosophy of Science*, D. Reidel, Boston.

Hund F. [1974]: *The History of Quantum Theory*, Harrap, London.

Inonu E., and E. Wigner [1952]: "Representations of the Galilei group", *Nuovo Cimento* **9**, 705-18.

Inonu E., and E. Wigner [1953]: "On the contraction of groups and their representations", *Proc. Nat. Acad. Sci.* **39**, 510-24.

Isham C. [1975]: "An introduction to quantum gravity", in *Quantum Gravity, an Oxford Symposium*, Clarendon Press, Oxford.

Iwanenko D., and L. Landau [1926]: "Zur Ableitung der Klein-Fockschen Gleichung", *Zeit. f. Phys.* **40**, 161-2.

Jammer M. [1966]: *The Conceptual Development of Quantum Mechanics*, Mc.Graw-Hill, New York.

Jammer M. [1974]: *The Philosophy of Quantum Mechanics*, Wiley-Interscience, New York.

Jauch J. [1964]: "The problem of measurement in quantum mechanics", *Helv. Phys. Acta* **37**, 293-316.

Jauch J. [1968]: *Foundations of Quantum Mechanics*, Addison-Wesley, Reading, Mass.

Jauch J. [1971]: "Foundations of quantum mechanics", *in* d'Espagnat [1971].

Jauch J., and C. Piron [1969]: "On the structure of quantal proposition systems", *Hel. Phys. Acta* **42**, 842-848.

Jeans J. [1919]: *Problems of Cosmogony and Stellar Dynamics*, CUP, Cambridge.

Jelinek F. [1968]: "BCS spin model, its thermodynamic representations and automorphisms", *Comm. Math. Phys.* **9**, 169-75.

Joos H. [1962]: "Zum Darstellungstheorie der inhomogenen Lorentzgruppe als Grundlage quantenmechanischer Kinematik", *Fort. d. Phys.* **10**, 65-146.

Jordan P. [1924]: "Zur Theorie der Quantenstrahlung", *Zeit. f. Phys.* **30**, 297

Jordan P. [1925]: "Über der thermische Gleichgewicht zwischen Quantenatomen und Hohlraumstrahlung", *Zeit. f. Phys.* **33**, 649.

Jordan P. [1927a] "Über die thermodynamische Gleichgewichtskonzentration der kosmischen Materie", *Zeit. f. Phys.* **41**, 711-17.

Jordan P. [1927b]: "Zur Quantenmechanik der Gasentartung", *Zeit. f. Phys.* **44**, 473-80.

Jordan P. [1927c] "Über Wellen und Korpuskeln in der Quantenmechanik", *Zeit. f. Phys.* **45**, 766-75.

Jordan P. [1929] "Der gegenwärtige Stand der Quantenelektrodynamiks", *Phys. Zeit.* **30**, 702.

Jordan P. [1932]: "Uber eine Klasse nicht assoziativer-hyperkomplexen Algebren", *Gott. Nachr.* 569-575.

Jordan P. [1933a]: "Uber Verallgemeinerungsmöglichkeiten des Formalismus der Quantenmechanik", *Gott. Nachr.* 209-17.

Jordan P. [1933b]: "Uber die Multiplikation quantenmechanischen Grossen", *Zeit. f. Phys.* **80**, 285-91.

Jordan P., and O. Klein [1927]: "Zum Mehrkörproblem der Quantentheorie", *Zeit. f. Phys.* **45**, 751-65.

Jordan P. and E. Wigner [1928]: "Über das Paulische Aquivalenzverbot", *Zeit. f. Phys.* **44**, 631-51.

Jordan P., and W. Pauli [1928]: "Zur Quantenelektrodynamik ladungsfreier Felder", *Zeit. f. Phys.* **47**, 151-73.

Jordan P., J. von Neumann, and E. Wigner [1934]: "On an algebraic generalization of the quantum mechanical formalism", *Ann. of Math.* **35**, 29-64.

Jost R. [1969](ed): *Local Quantum Theory, Proceedings of the International School of Physics "Enrico Fermi"*, Academic Press, New York.

Kadison R. [1967]: "Lectures on operator algebras" *in* Lurcat [1967 p.41-87].

Kakatuni S., and G. Mackey [1944]: "Two characterizations of real Hilbert space", *Ann. of Math.* **45**, 50-8.

Kakatuni S., and G. Mackey [1946]: "Ring and lattice characterisations of complex Hilbert space", *Bull. Amer. Math. Soc.* **52**, 727-33.

Kaluza T. [1921]: "Zum Unitatsproblem der Physik", *Sitz. d. Akad. Wissen. z. Berlin* 966-72.

Kaplanski I. [1951]: "Projections in Banach algebras", *Ann. of Math.* **53**, 235-249.

Kaplanski I. [1955]: "Any orthocomplemented complete modular lattice is a continuous geometry", *Ann. of Math.* **61**, 524-41.

Kastler D. [1964]: "A C^*-algebra approach to field theory", *in* Martin and Segal [1964 p.179-91].

Kastler D. [1967]: "Topics in the algebraic approach to field theory" *in* Lurcat [1967].

Kastler D., and M. Takesaki [1979]: "Group duality and the Kubo-Martin-Schwinger condition", *Comm. Math. Phys.* **70**, 193-212.

Kastler D., and M. Takesaki [1982]: "Group duality and the Kubo-Martin-Schwinger Condition", *Comm. Math. Phys.* **85**, 155-76.

Kecs W., and P. Tesdorescu [1974]: *Applications of the Theory of Distributions in Mechanics*, Abacus Press, Tunbridge Wells.

Kemmer N. [1938]: "Quantum theory of Einstein-Bose particles and nucliear interactions", *Proc. Roy. Soc. A* **166**, 127-53.

Kemmer N. [1939]: "The particle aspect of the meson theory", *Proc. Roy. Soc. London A* **173**, 91-116.

Kilmi-ster C. [1987] (ed): *Schrödinger: Centenary Celebration of a Polymath*, CUP, Cambridge.

Klein M. [1964]: "Einstein and the wave-particle duality", *The Natural Philosopher* **3**, 1-49.

Klein O. [1926a]: "Quantentheorie und fünfdimensionale Relativitatstheorie", *Zeit. f. Phys.* **37**, 895-906.

Klein O. [1926b]: "The atomicity of electricity as a quantum theoretic law", *Nature* **118**, 516.

Klein O. [1929]: "Die Reflexion von Elektronen an einem Pontialsprung nach der relativistischen Dynamik von Dirac", *Zeit. f. Phys.* **53**, 157-65.

Klein O., and Y. Nishina [1929]: "Über die Streuung von Strahlung durch freie Elektronen nach der neuen relativistischen Quantendynamik von Dirac", *Zeit. f. Phys.* **52**, 853-68.

Knight [1961]: "Strict localization in quantum field theory", *J. Math. Phys.* **2**, 459.

Kochen S., and E. Specker [1967]: "The problem of hidden variables in quantum mechanics", *J. Math. and Mech.* **17**, 59-87.

Koopman B. [1931]: "Hamiltonian systems and transformations in Hilbert spaces" *Proc. Nat. Acad. Sci.* **17**, 315-18.

Kragh H. [1981]: "The genesis of Dirac's relativistic theory of electrons", *Archive for History of Exact Sciences* **24**, 31-67.

Kramers H. [1937]: "The use of charge-conjugated wave functions in the hole-theory of the electron", *Proc. Kgl. Ned. Acad. Wet.* **40**, 814-23.

Kramers H., and W. Heisenberg [1925]:"Über die Streuung von Strahlung durch Atome", *Zeit. f. Phys.* **31**, 681-707, *English translation in* van der Waerden [1967 p.223-251].

Kubo R. [1957]: "Statistical mechanical theory of irreversible processes", *J. Phys. Soc. (Japan)* **12**, 570-86.

Kuhn T. [1978]: *Black-Body Theory and the Quantum Discontinuity*, Clarendon Press, Oxford.

Lahti P., and P. Mittelstaedt [1985]: *Symposium on the Foundations of Modern Physics*, World Scientific Publishing Co., New York.

Landau L., and R. Pieirls [1931]: "Erweiterung des Unbestimmtheitsprinzip für die Relativistische Quantentheorie", *Zeit. f. Phys.* **69**, 57.

Lanford O. and D. Ruelle [1969]: "Observables at infinity and states with short-range correlations in statistical mechanics", *Commun. Math. Phys.* **13**, 194-215.

Lapedes A. [1978]: "Bogoliubov transformations, propagators, and the Hawking effect", *J. Math. Phys.* **19**, 2289-93.

Laporte O. and G. Uhlenbeck [1931]: "Application of the spinor analysis to the Maxwell and Dirac equations", *Phys. Rev.* **37**, 1380-1397.

Lebowitz J. [1968]: "Statistical mechanics - a review of selected rigorous results", *Ann. Rev. Phys. Chem.* **19**, 389-418.

Lee T., and C. Yang [1952]: "Statistical theory of equations of state and phase transitions; II latice gas and Ising model", *Phys. Rev.* **87**, 401-19.

Leggett A.J. [1985]: "Quantum mechanics versus macroscopic realism: is there flux there when nobody looks?", *Phys. Rev. Lett.* **54**, 857.

Lewis D., and L. Thomas [1975]: "On the existence of a class of stationary quantum stochastic processes", *Ann. Inst. H. Poincare* **A22**, 241-8.

Lichnerowicz A. [1976]: *Global Theory of Connections and Holonomy Groups*, Noordhoff, Leyden.

Loomis L. [1947]: "On the representation of σ-complete Boolean algebras", *Bull. Amer. Math. Soc.* **53**, 757-760.

Lurçat F. [1967]: *Cargèse Lectures in Theoretical Physics*, Gordan and Breach, New York.

Mackey G. [1942]: "Isomorphisms of normed linear spaces", *Ann. of Math.* **43**, 244-260.

Mackey G [1949a]: "A theory of Stone and von Neumann", *Duke Math. J.* **16**, 313-325.

Mackey G. [1949b] "Imprimitivity for representations of locally compact groups 1", *Proc. Nat. Acad. Sci.* **35**, 537- 545.

Mackey G. [1951]: "Induced representations of groups", *Amer. J. of Math.* **73**, 576-592.

Mackey G. [1952]: "Induced representations of locally compact groups 1", *Ann. of Math.* **55**, 101-139.

Mackey G. [1953]: "Induced representations of locally compact groups II. The Frobenius reciprocity theorem" *Ann. of Math.* **58**, 193-221.

Mackey G. [1957]: "Borel structures in groups and their duals" *Trans. Amer. Math. Soc.* **85**, 134-165.

Mackey G. [1958]: "Unitary representations of group extensions, 1" *Acta Math.* **99**, 265-311.

Mackey G. [1963a]: "Infinite-dimensional group representations", *Bull. Am. Math. Soc.* **69**, 628-86.

Mackey G. [1963b]: "Group representations in Hilbert space" *in* Segal [1963 p.113-30].

Mackey G. [1963c]: *Mathematical Foundations of Quantum Mechanics*, Benjamin, N. York.

Mackey G. [1980]: "Harmonic analysis as the exploitation of symmetry - a historical survey", *Bull. Amer. Math. Soc. (New Ser.)* **3**, 543-698.

Mackinnon E. [1976]: "De Broglie's thesis: a critical retrospective", *Am. J. Phys.* **11**, 1047-55.

MacLaren M. [1964]: "Atomic orthocomplemented lattices", *Pacific J. of Math.* **14**, 597-612.

Majorana E. [1932]: "Teoria relativistica di particelle con momento intrinseco arbitrario", *Nuovo Cimento* **9**, 335-344.

Margenau H. [1934]: "On the application of many-valued systems of logic to physics", *Philosophy of Science* **1**, 118-21.

Martellini M. [1982]: "Quantum field theory on curved space-time: an axiomatic approach", *Il Nuovo Cimento* **67**, 305-55.

Martin W. and J. Schwinger [1959]: "Theory of many-particle systems: 1", *Phys. Rev.* **115**, 1342-73.

Martin W. and I. Segal [1964] (eds.): *Analysis in Function Space*, the MIT Press, Cambridge, Mass.

Massen H. [1982]: "A quantum field acting as a heat bath" *Phys. Lett.* **91A**, 107-11.

Massen H. [1984]: "Return to thermal equilibrium by the solution of a quantum Langevin equation", *J. Stat. Phys.* **34**, 239-61.

Mautner F. [1950]: "Unitary representations of locally compact groups I". *Ann. of Math.* **51**, 1-25, II, 528-556.

Mehra J. [1973]: *Symposium on the Development of the Physicist's Conception of Nature in the 20th. Century*, Dordrecht and Co.

Moller C. [1949]: "On the definition of the centre of gravity of an arbitrary closed system in the theory of relativity", *Comm. Dublin Inst. Adv. St. A, No. 5*, p.3-42.

Murray F., and J. von Neumann [1936]: "On rings of operators" *Ann. of Math.* **37**, 116-129.

Nelson E. [1964]: "Feynman integrals and the Schrödinger equations", *J. Math. Phys.* **5**, 332-43.

Nelson E. [1973]: "Construction of quantum fields from Markoff fields", *J. Func. Anal.* **12**, 97-112.

Newton T. and E. Wigner [1949]: "Localized states for elementary systems", *Rev. Mod. Phys.* **21**, 400-406.

Noll W. [1958]: "A mathematical theory of the mechanical behaviour of continuous media", *Arch. Rat. Mech. Anal.* **2**, 197-226.

Novozhilov Y. [1975]: *Introduction to Elementary Particle Theory*, Pergamon Press.

Ornstein L., and F. Zernike [1919]: *Proc. K. Ak. Amsterdam* **28**, 280

Pais A. [1986]: *Inward Bound, of Matter and Forces in the Physical World*, Clarendon Press, Oxford.

Pauli W. [1933a]: "Die allgemeinen Prinzipien der Wellenmechanik", *Handbuch der Physik*, H. Geiger and K. Scheel (eds.), 2nd. ed., Vol. 24, Springer, Berlin. *This aritcle appears substantially unchanged as* W. Pauli, *General Principles of Quantum Mechanics*, Springer-Verlag, Berlin, 1980.

Pauli W. [1933b] "Einige die Quantenmechanik betreffenden Erkundigungsfragen", *Zeit. f. Phys.* **80**, 573-86.

Pauli W. [1979]: *W. Pauli, Scientific Correspondence*, Springer-Verlag, New York.

Pauli W., and M. Fierz [1939]: "Über relativistische Feldgleichungen von Teilchen mit beliebegen Spin in elektromagnetischen Feld", *Helv. Phys. Acta* **12**, 297-300.

Pauli W., and V. Weisskopf [1934]: "Über die Quantisierung der skalaren relativistischen Wellengleichung", *Helv. Phys. Acta* **7**, 709-731.

Williams L. Pierce [1965]: *Micheal Faraday, a Biography*, London.

Planck M. [1921]: "Absolute Entropie und chemische Konstante", *Ann. der Phys.* **66**, 365-72.

Planck M. [1925a]: "Zur Frage des Quantelung einatomiger Gase", *Sitz. d. Preus. Akad. d. Wiss.* 49-57.

Planck M. [1925b]: "Über die Statistische Entropiedefinition", *Sitz. d. Preus. Akad. d. Wiss.* 442-51.

Plymen R. [1968a]: "C^*-algebras and Mackey's axioms", *Commuun. Math. Phys.* **8**, 132-146.

Plymen R. [1968b]: "A modification of Piron's axioms", *Helv. Phys. Acta* **41**, 69-74

Pons J. [1983]: "World lines, Poincare realizations and predictive relativistic mechanics", *Ann. of Phys.* **148**, 192-213.

Poole J. [1968]: "Baer * semigroups and the logic of quantum mechanics", *Comm. Math. Phys.* **9**, 118-41, 212.

Popper K. [1968]: "Birkhoff and von Neumann's interpretation of quantum mechanics", *Nature* **219**, 682-685.

Price W., and S. Chissick [1977] (eds.): *The Uncertainty Principle and Foundations of Quantum Mechanics, a Fifty Years' Survey*, Wiley-Interscience, London.

Primas H. [1980]: "Foundations of theoretical chemistry", *in* Woolley [1980].

Primas H. [1981]: *Chemistry, Quantum Mechanics and Reductionism, Lecture Notes in Chemistry 24*, Springer-Verlag, Berlin.

Proca A. [1936]: "Sur la theorie ondulatoire des electrons positifs et negatifs", *J. de Phys. et de Radium* **7**, 347-53.

Prosperi G. [1971]: "Macroscopic physics and the problem of measurement in quantum mechanics", *in* d'Espagnat [1971 p.97-124].

Pryce M. [1948]: "The mass-centre in the restricted theory of relativity and its connexion with the quantum theory of elementary particles", *Proc. Roy. Soc. London A* **195**, 62-81.

Redhead M. [1981]: "Experimental tests of the sum rule", *Phil. of Science* **48**, 50-64.

Redhead M. [1983]: "Quantum field theory for philosophers", *Proceedings of the 1982 Biennial Meeting of the Philosophy of Science Assoociation Vol. 2.*, p.57-99.

Redhead M. [1988]: "A philosopher looks at quantum field theory", *in* Brown and Harre [1988 p.9-24].

Reed M., and B. Simon [1975]: *Methods of Mordern Mathematical Physics, Volume 2. Fourier Analysis, Self-Adjointness*, Academic Press, New York.

Reeh H., and S. Schlieder [1962]: "Über den Zerfall der Feldoperator algebra im Falle einer Vakumentartung", *Nuovo Cimento* **26**, 32-42.

Rohrlich F. [1985]: "Schrödinger's criticism of quantum mechanics - fifty years later" *in* Lahti and Mittelstaedt [1985 p.555-72].

Rose M. [1957]: *Elementary Theory of Angular Momentum*, John Wiley, New York.

Rosenfeld L. [1963]: "On quantization of fields", *Nuc. Phys.* **40**, 353-56.

Ruelle D. [1962]: "On the asymptotic condition in quantum field theory", *Hel. Phys. Acta* **35**, 147-163.

Ruelle D. [1965]: "Correlation functionals", *J. Math. Phys.* **6**, 201-20.

Ruelle D. [1966]: "States of Physical Systems", *Commun. Math. Phys.* **3**, 133-150.

Ruelle D. [1967]: "A variational formulation of equilibrium statistical mechanics and the Gibbs phase rule", *Comm. Math. Phys.* **5**, 324.

Ruelle D. [1969]: *Statistical Mechanics*, Benjamin, Reading, Mass.

Sakata and Taketani [1940]: "On the wave equation of meson", *Proc. Math. Phys. Soc., Japan* **22**, 757-70.

Sackur O. [1911]: "Die Anwendung der kinetistischen Theorie der Gase auf chemische Probleme", *Ann. der Phys.* **36**, 958

Salam A., and E. Wigner [1972] (eds.): *Aspects of Quantum Theory*, Cambridge University Press, Cambridge, Mass.

Saunders S. [1988]: "The algebraic approach to quantum field theory" *in* Brown and Harre [1988 p.149-86].

Saunders S., H. Brown, and R. Harre [1989]: *The Philosophy of the Vacuum*, Clarendon Press, Oxford.

Schiff L. [1968]: *Quantum Mechanics*, 3rd. Ed., McGraw-Hill, Tokyo.

Schlick P. [1970]:*Albert Einstein Philosopher-Scientist*, 3rd. Ed. Open Court, Illinois.

Schrieffer J. [1964]: *Theory of Superconductivity*, Benjamin, Reading, Mass.

Schrödinger E. [1925]: "Bemerkungen über die Statistische Entropiedefinition bein idealen Gas", *Sitz. d. Preus. Akad. d. Wiss. Berlinner Berichte* 434-41.

Schrödinger E. [1926a]: "Zur Einsteinschen Gastheorie", *Phys. Zeit.* **27**, 95-101.

Schrödinger E. [1926b]: "Quantisierung als Eigenwertproblem IV", *Ann. der Phys.* **81**, 109-139. *English translation in* Schrödinger [1928 p.102-23].

Schrödinger E. [1927]: "Der Energieimpulssatz der Materiewellen", *Ann. der Phys.* **82**, 265-72. *English translation in* Schrödinger [1928 p.130-6].

Schrödinger E. [1928]: *Collected Papers on Wave Mechanics*, Blackie and Son, London.

Schrödinger E. [1930]: "Über die kraftefreie Bewegung in der relativistischen Quantenmechanik", *Sitz. d. Preus. Akad. d. Wiss.* 418-28.

Schrödinger E. [1932]: "Sur la theorie relativiste de l'electron et l'interpretation de la mecanique quantique", *Ann Inn. Henri Poincare* **2**, 269-310.

Schweber S. [1961]: *An Introduction to Relativistic Quantum Field Theory*, Harper and Row, New York.

Schweber S. [1986]: "Feynman and the visualization of space-time processes", *Rev. Mod. Phys.* **58**, 449-508.

Sciama D., P. Candelas, and D. Deutsch [1981]: "Quantum field theory, horizons and thermodynamics", *Adv. Phys.* **30**, 327-66.

Scott W. [1967] *Erwin Schrödinger: an Introduction to his Writings*, University of Mitchigan Press, Amherst.

Segal I. [1947a]: "Irreducible representations of operator algebras", *Bull. Amer. Math. Soc.* 73-88.

Segal I. [1947a]: "Postulates for general quantum mechanics" *Ann. of Math.* **48**, 930-948.

Segal I. [1951]: "A class of operator algebras which are determined by groups", *Duke Math. J.* **18**, 221-265.

Segal I. [1954]: "Abstract probability spaces and a theorem of Kolmogoroff", *Amer. J. Math.* **76**, 721-32

Segal I. [1956a]: "Tensor algebras over Hilbert spaces: 1", *Trans. Amer. Math. Soc.* **81**, 106-134.

Segal I. [1956b]: "Tensor algebras over Hilbert spaces: 2", *Ann. of Math.* **63**, 160-75.

Segal I. [1959a]: "Foundations of the theory of dynamical systems of infinitely many degrees of freedom I", *Mat. Fys. Medd. Dan. Vid. Selsk.* **31**, No. 2.

Segal I. [1959b]: "The mathematical meaning of operationalism in quantum mechanics", *Symposium on the Axiomatic Method*, ed. L.Henkin, P.Suppes, and A. Tarski, North Holland, Amsterdam, p.341-352.

Segal I. [1961]: "Foundations of the theory of dynamical systems of infinitely many degrees of freedom II", *Canadian J. Math.* **13**, 1-18.

Segal I. [1962]: "Foundations of the theory of dynamical systems of infinitely many degrees of freedom III", *J. Math.* **6**, 500-23.

Segal I. [1963]: "Mathematical problems of relativistic physics", *American Mathematical Society*, Providence, R.I.

Segal I. [1964]: "Quantum fields and analysis in the solution manifolds of differential equations", *in* Martin and Segal [1964].

Segal I. [1967a]: "Representations of the canonical commutation relationships" *in* Lurcat [1967].

Segal I. [1967b]: "Notes towards the construction of nonlinear relativistic quantum fields: 1; the Hamiltonian in two space-time dimensions as the generator of a C^*-automorphism group", *Proc. Nat. Acad. Sci.* **57**, 1178-83.

Segal I. [1970]: "Construction of nonlinear local quantum processes: 1, *Ann. of Math.* **92**, 462-81.

Segal I. [1974]: "Symplectic structures and the quantization problem for wave equations", in *Symposia mathematica*, Vol. 14, Academic Press, London.

Segal I. and R. Goodman [1965]: "Anti-locality of certain Lorentz-invariant operators", *J. Math. and Mech.* **14**, 629-638.

Serwer D. [1977]: "Unmechanischer Zwang: Pauli, Heisenberg, and the rejection of the mechanical atom 1923-1925", *Historical Studies in the Physical Sciences* **8**, 189-256.

Sewell G. [1986]: *Quantum Theory of Collective Phenomena*, Clarendon Press, Oxford.

Shale D., and W. Stinespring [1964]: "States of the Clifford algebra", *Ann. of Math.* **80**, 365-381.

Simon B. [1974]: *The $P(\phi)_2$ Euclidean (Quantum) Field Theory*, Princeton University Press, Princeton.

Stapp H. [1962]: "Derivation of the CPT theorem and the connection between spin and statistics from postulates of the S-matrix theory", *Phys. Rev.* **125**, 2139-67.

Stern O. [1925]: "Über das Gleichgewicht zwischen Materie und Strahlung", *Zeit. f. Elektrochem.* **31**, 448-49.

Stern O. [1926]: "Über die Umwandlung von Atomen in Strahlung", *Zeit. f. Phys. Chem.* **120**, 60-2.

Sternberg S. [1964]: *Lectures on Differential Geometry*, Prentice-Hall, Englewood Cliffs, N.J.

Stormer E. [1966]: "Jordan algebras of type I", *Acta Mathematica* **115**, 165-84.

Stone M. [1930a]: "Linear transformations in Hilbert space. 1. Geometrical aspects", *Proc. Nat. Acad. Sci.* **15**, 198-200.

Stone M. [1930b]: "Linear transformations in Hilbert space. 3. Operational methods and group theory", *Proc. Nat. Acad. Sci.* **16**, 172-175.

Stone M. [1932a]: *Linear Transformations in Hilbert Space and their Applications to Analysis*, American Mathematical Society Colloquium Publications, Providence, R.I.

Stone M. [1932b]: "On one -parameter unitary groups in Hilbert space", *Ann. of Math.* **33**, 643-648.

Stone M. [1934]: "Boolean algebras and their application to topology", *Proc. Nat. Acad. Sci.* **20**, 197-202.

Stone M. [1936]: "The theory of representations of Boolean algebras", *Trans. Am. Math. Soc.* **40**, 37-111.

Stoops R. [1962]: *La Theorie Quantique des Champs, 12th Solvay Conference*, Interscience, New York.

Streater R. [1972] (ed.): *The Mathematics of Contemporary Physics*, Academic Press, New York.

Streater R. [1975]: "Outline of axiomatic relativistic quantum field theory", *Rep. Prog. Phys.* **38**, 771-846.

Streater R., and A. Wightman [1964]: *TCP, Spin and Statistics, and All That*, Benjamin, New York.

Streit L. [1968]: "A generalization of Haag's theorem", preprint, Pittsburgh.

Stueckelberg E. [1960]: "Quantum theory in real Hilbert space", *Helv. Phys. Acta* **33**, 727-752,

Streater R. [1988]: "Why should anyone wish to axiomatize quantum field theory?" in Brown and Harré [1988 p. 135-148].

Stueckelberg E., and M. Guenin [1961a]: "Quantum theory in real Hilbert space", *Helv. Phys. Acta* **34**, 621-628.

Stueckelberg E., and M. Guenin [1962]: ""Quantum theory in real Hilbert space", *Helv. Phys. Acta* **35**, 673-695.

Stueckelberg E., M. Guenin, C. Piron, and H. Ruegg [1961b]: "Quantum theory in real Hilbert space", *Helv. Phys. Acta* **34**, 675-698.

Sudarshan E. [1963]: "Equivalence of semiclassical and quantum mechanical descriptions of statistical light beams", *Phys. Rev. Lett.* **10**, 277-9.

Sudarshan E. and N. Mukunda [1974]: *Classical Dynamics: A Modern Perspective*, Wiley-Interscience, New York.

Symanzik [1964]: "Applications of functional intergration to Euclidean quantum field theory" *in* Martin and Segal [1964 p.197-213].

Takeda Z. [1955]: "Inductive limit and infinite direct product of operator algebras", *Tohoku. Math. J.* **7**, 68-86.

Takesaki M. [1970]: "Disjointness of the KMS states of different termperatures", *Comm. Math. Phys.* **17**, 33-41.

Teller P. [1989]: "How to have an honest vacuum" *in* Saunders *et al* [1989].

Tetrode H. [1912]: "Die chemische Konstante der Gase und das elementere Wirkungsquantum". *Ann. der Phys.* **38**, 434-42 *corrected in* **39**, 255-6.

Thirring W. [1979]: *A Course in Mathematical Physics 3: Quantum Mechanics of Atoms and Molecules*, Springer-Verlag, Berlin.

Thirring W. and A.Wehrl [1967]: "On the mathematical structure of the BCS model", *Commun. of Math. Phys.* **4**, 303-314.

Tomonaga S. [1962]: *Quantum Mechanics Vol. I, II*, Amsterdam.

Unruh W. [1974]: "Second quantization in the Kerr metric", *Phys. Rev. D* **10**, 3194-205.

Unruh W. [1976]: "Notes on black-hole evaporation", *Phys. Rev. D* **14**, 870-92.

van der Waerden [1929]: "Spinoranalyse", *Nachrichten von der Gessellshaft der Wissen. zu Gottingen, Math.-Phys. K.*, 100-109.

Van der Waerden B. [1967]: *Sources of Quantum Mechanics*, North-Holland, Amsterdam.

Van Fleck J. [1924]: "The absorption of radiation by multiply periodic orbits, and its relation to the correspondence principle and the Rayleigh-Jeans law", *Phys. Rev.* **24**, 330-365.

Van Hove L. [1951]: "Sur l'operateur Hamiltonien de deux champs quantifiés en interaction", *Acad. Roy. Belg. Bull. Classe Sci.* **37**, 1055-72.

Van Hove L. [1952]: "Les difficultés de divergences pour un modèle particulier de champ quantifié", *Physica* **18**, 145-59.

Varadarajan V. [1962]: "Probability in physics and a theorem on simultaneous probability" *Commun. in Pure App. Math.* **15**, 189-217. Corrected *loc cit* 1965 **18**.

Varadarajan V. [1968]: *Geometry of Quantum Theory*, Volume 1, Van Nostrand, Princeton N.J.

Varadarajan V. [1970]: *Geometry of Quantum Theory*, Volume 2, Van Nostrand, Princeton N.J.

von Neumann J. [1931]: "Die Eindeutigkeit der Schrödingerschen Operatoren", *Math. Ann.* **104**, 570-8.

von Neumann J. [1932]: *Grundlagen der Quantenmechanik*, Springer, Berlin; *English translation by* R. T. Beyer, Princeton Universtity Press, Princeton, N.J.

von Neumann J. [1936]: "On an algebraic generalization of the quantum mechanical formalism (Part 1)", *Mat. Sborn.* **1**, 415-484.

von Neumann J. [1949]: "On rings of operators. Reduction theory", *Ann. of Math.* **50**, 401-485.

Waller I. [1930]: "Die Streuung von Strahlung durch gebundene und freie Elektronen nach der Diracschen relatistischen Mechanik", *Zeit. f. Phys.* **61**, 837-51.

Weil A. [1938]: *Integration dans les groupes topolgiques et ses appliations a l'analyse*, Actualités Scientifiques et Industrielles, No. 869, Paris.

Weinberg S. [1964]: "Feynman rules for any spin", *Phys. Rev. D* **133**, 1318-1332.

Weiner C. [1977] (ed): *History of 20th Century Physics, Proceedings of the International School of Physics "Enrico Fermi", Course LVII*, Academic Press, New York.

Weingard R. [1988]: "Virtual particles and the interpretation of quantum field theory", *in* Brown and Harre [1988 p.43-58].

Wentzel [1949]: *Quantum Theory of Fields*, Interscience, New York.

Wessels L. [1979]: "Schrödinger's route to wave mechanics". *Studies in the Hist. and Phil. of Sci.* **10**, 311-40.

Weyl H. [1928]: *Gruppentheorie und Quantenmechanik*, S. Hirzel, Leipzig. *English translation* Methuen, London, 1931.

Weyl H. [1946]: *The Classical Groups*, Princeton University Press, Princeton, N.J.

Whitten-Wolfe B., and G. Emch [1976]: "A mechanical quantum measuring process", *Helv. Phys. Acta* **49**, 45-55.

Wightman A., and S. Schweber [1955]: "Configuration space methods in relativistic quantum field theory, 1", *Phys. Rev.* **98**, 812-837.

Wightman A. [1962]: "Localizability of quantum mechanical systems", *Rev. Mod. Phys.* **34**, 845-872.

Wightman A. [1972] "The Dirac equation", *in* Salam and Wigner [1972 p.95-115].

Wigner E. [1931]: *Gruppentheorie und ihre Anwendung*, Wieweg, Braunschweig; English translation by J. Griffin, Academic Press, New York, 1959.

Wigner E. [1935]: "Symmetry relations in various physical problems", *Bull. Amer. Math. Soc.* **41**, 306.

Wigner E. [1939]: "On unitary representations of the inhomogeneous Lorentz group", *Ann. of Math.* **40**, 149-204.

Wigner E. [1947]: "Relativistische Wellengleichungen", *Zeit. f. Phys.* **124**, 665-684.

Woodhouse N. [1980]: *Geometric Quantization*, Clarendon Press, Oxford.

Woolley R. [1980]: *Quantum Dynamics of Molecules*, Nato Advanced Study Institutes Series B, Plenum Press, New York.

Zawirski Z. [1932]: "Les logiques nouvelles et le champ de leur application", *Revue de Metaphysique et de Morale* 39, 503-19.

Zurek W. [1981]: "Pointer basis of quantum mechanics: into what mixture does the wave packet collapse?", *Phys. Rev. D* **24**, 1516-25.

Zwicky F. [1933]: "On a new type of reasoning and some of its possible consequences", *Phys. Rev.* **43**, 1031-33.